Quantitative heat-kernel estimates for diffusions with distributional drift

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Abstract

We consider the stochastic differential equation on $\mathbb{R}^d$ given by

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}B_t,$$

where B is a Brownian motion and b is considered to be a distribution of regularity $> -\frac{1}{2}$. We show that the martingale solution of the SDE has a transition kernel Γ_t and prove upper and lower heat-kernel estimates for Γ_t with explicit dependence on t and the norm of b.

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1 Introduction and main results

In this paper we consider the stochastic differential equation on $\mathbb{R}^d$ given by

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}B_t, \tag{1}$$

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where B is a Brownian motion and b is a distribution of regularity $> -\frac{1}{2}$. Such *singular diffusions* (diffusions with distributional drift) appear as models for stochastic processes in random media (then b would also be random, but independent of B), for example in [4, 6, 5]. They also appear as "stochastic characteristics" in Feynman-Kac type representations of singular SPDEs, for example in [13, 5, 17]. In non-singular SPDEs, the stochastic characteristics would be formulated in terms of the Brownian motion, and they may be useful tools to infer information about the long time behavior of the SPDE. For example, the asymptotic behavior of the total mass of the parabolic Anderson model is typically derived via the Feynman-Kac formula [16], and for that purpose it is important that we understand the Brownian motion and its transition probabilities very well. When studying singular variants of the parabolic Anderson model, where the Brownian motion in the Feynman-Kac representation is replaced by a singular diffusion, we thus need to understand the transition probabilities of this singular diffusion. Moreover, since we are interested in the long time behavior, we need quantitative control of the transition probabilities on arbitrarily long time intervals. This motivates our present work.

We show that the solution to (1) possesses a transition kernel $\Gamma_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ for all $t > 0$. This means that under the measure $\mathbb{P}_x$ such that $X_0 = x$ we have for all $\phi \in C_{\mathrm{b}}(\mathbb{R}^d)$

$$\mathbb{E}_x[\phi(X_t)] = \int_{\mathbb{R}^d} \phi(y)\Gamma_t(x,y)\,\mathrm{d}y.$$

The following theorem represents the main result of our paper, in which we show that the above transition kernel satisfies heat-kernel estimates.

For any Banach space $\mathfrak{X}$ and $t > 0$ we write $\|\cdot\|_{C_t\mathfrak{X}}$ for the norm on $C([0,t],\mathfrak{X})$, which is defined for $f \in C([0,t],\mathfrak{X})$ by

$$\|f\|_{C_t\mathfrak{X}} = \sup_{s\in[0,t]} \|f(s)\|_{\mathfrak{X}}.$$

$\Delta_{-1}b$ denotes the first Littlewood-Payley block and $\Delta_{\geq 0}b$ the sum of the positive Littlewood-Payley blocks (see Section 1.2). $B^s_{p,q}$ denotes a Besov space, see [2].

Theorem 1.1. *Let $\alpha \in (0,\frac{1}{2})$ and $c > 1$. There exist a $C > 1$ and a $\kappa \in (0,1)$ such that for all $b = (b_t)_{t\geq 0} \in C([0,\infty), B^{-\alpha}_{\infty,1}(\mathbb{R}^d,\mathbb{R}^d))$, $\mu \in \mathbb{N}_0^d$ with $|\mu| \leq 1$, and for all $t > 0$, $x, y \in \mathbb{R}^d$:*

$$|\partial_x^\mu \Gamma_t(x,y)| \leq C \exp\left(Ct\Big[\|\Delta_{-1}b\|^2_{C_tL^\infty} + \|\Delta_{\geq 0}b\|^{\frac{2}{1-\alpha}}_{C_tB^{-\alpha}_{\infty,1}}\Big]\right)(t^{-\frac{|\mu|}{2}} \vee 1)p(ct, x-y), \quad (2)$$

$$\Gamma_t(x,y) \geq \frac{1}{C}\exp\left(-Ct\Big[\|\Delta_{-1}b\|^2_{C_tL^\infty} + \|\Delta_{\geq 0}b\|^{\frac{2}{1-\alpha}}_{C_tB^{-\alpha}_{\infty,1}}\Big]\right) p(\kappa t, x-y), \quad (3)$$

where $p(t,x) = (2\pi t)^{-\frac{d}{2}}\mathrm{e}^{-|x|^2/2t}$ is the standard Gaussian kernel.

As a corollary, we obtain the following estimate on the escape probability of the diffusion X to leave a ball.

Corollary 1.2. *Let $\alpha \in (0, \frac{1}{2})$. There exists a $C > 0$ such that for all $b \in C([0,\infty), B^{-\alpha}_{\infty,1}(\mathbb{R}^d, \mathbb{R}^d))$, $x \in \mathbb{R}^d$, $K > 0$ and $T \geq 1$, and for X solving* (1) *with $\mathbb{P}_x(X_0 = x) = 1$:*

$$\mathbb{P}_x\Big(\sup_{t\in[0,T]} |X_t - x| \geq K\Big) \\ \leq C \exp\Big(CT\Big[\|\Delta_{-1}b\|^2_{C_T L^\infty} + \|\Delta_{\geq 0}b\|^{\frac{2}{1-\alpha}}_{C_T B^{-\alpha}_{\infty,1}}\Big]\Big) \exp\Big(-\frac{K^2}{CT}\Big) \tag{4}$$

Remark 1.3. At least for constant b the heat-kernel estimates are sharp: If $\lambda \in \mathbb{R}^d$ and $b = \lambda$, then $\Gamma_t(x,y) = p(t, y - x - \lambda t)$ and a simple computation shows that $\sup_{x\in\mathbb{R}^d} \frac{p(t,x-\lambda t)}{p(ct,x)} = c^{\frac{d}{2}} e^{\frac{1}{2(c-1)}t\lambda^2}$ and $\inf_{x\in\mathbb{R}^d} \frac{p(t,x-\lambda t)}{p(\kappa t,x)} = \kappa^{\frac{d}{2}} e^{-\frac{1}{2(1-\kappa)}t\lambda^2}$. Since in that case $\Delta_{\geq 0} b = 0$, this corresponds exactly to our bounds (2) and (3) (for $\mu = 0$).

Indeed, if X_t is the solution of

$$\mathrm{d}X_t = \lambda\,\mathrm{d}t + \mathrm{d}B_t,$$

then $X_t = X_0 + \lambda t + B_t$ and thus for $\mathbb{P}$ being the probability such that B_t is a standard Brownian motion and $\mathbb{P}_x$ the probability under which X satisfies the SDE with $X_0 = x$ a.s.,

$$\mathbb{P}_x(X_t \in A) = \mathbb{P}(x + \lambda t + B_t \in A) = \int_{A-x-\lambda t} p(t,y)\,\mathrm{d}y = \int_A p(t, y - x - \lambda t)\,\mathrm{d}y.$$

We have

$$\frac{\Gamma_t(0,y)}{p(ct,y)} = \frac{p(t, y-\lambda t)}{p(ct,y)} = c^{\frac{d}{2}}\left[\exp\left(\frac{|y|^2}{2ct} - \frac{|y-\lambda t|^2}{2t}\right)\right].$$

We calculate

$$\sup_{y\in\mathbb{R}^d} |y|^2 - c|y-\lambda t|^2.$$

The function over which we take the derivative is concave, so we calculate the point at which the gradient equals zero. The derivative equals $\partial_i |y|^2 - c|y - \lambda t|^2 = 2y_i - 2c(y_i - \lambda_i t)$. This equals zero for $y_i = \frac{c\lambda_i t}{c-1}$ and thus

$$\sup_{y\in\mathbb{R}^d} |y|^2 - c|y-\lambda t|^2 = \frac{c-1}{c}|y|^2 = \frac{c}{c-1}|\lambda|^2 t^2,$$

and thus

$$\sup_{y\in\mathbb{R}^d} \frac{\Gamma_t(0,y)}{p(ct,y)} = c^{\frac{d}{2}} \exp\left(\frac{|\lambda|^2 t}{2(c-1)}\right).$$

For the lower bound we take instead of the supremum the infimum and replace c by κ, which makes the function $|y|^2 - \kappa|y-\lambda t|^2$ convex. Then we find

$$\inf_{y\in\mathbb{R}^d} \frac{\Gamma_t(0,y)}{p(\kappa t,y)} = \kappa^{\frac{d}{2}} \exp\left(-\frac{|\lambda|^2 t}{2(1-\kappa)}\right).$$

Remark 1.4. As we consider a time inhomogeneous drift, we could have also formulated the heat-kernel estimates for $\Gamma_{s,t}$ (with $0 \leq s < t$), which is the transition kernel from time s to time t: If $\mathbb{P}_{s,x}$ is the probability measure under which $X_s = x$ and (1) holds (for $t > s$), then $\mathbb{E}_{s,x}[\varphi(X_t)] = \int_{\mathbb{R}^d} \varphi(y)\Gamma_{s,t}(x,y)\,\mathrm{d}y$. However, to simplify notation we only consider the case $s = 0$ and we write Γ_t for $\Gamma_{0,t}$. The heat-kernel estimates for $\Gamma_{s,t}$ follow by applying Theorem 1.1 with $b'_t = b_{t+s}$, $t \geq 0$.

1.1 Literature

Diffusions with a distributional drift were first considered by Bass and Chen [3] and Flandoli, Russo and Wolf [8], both in the one-dimensional time-homogeneous setting. More recently, Delarue and Diel [6] used Hairer's rough path approach to singular SPDEs [14, 15] to extend the results of [8] to the time-inhomogeneous case, and they applied this to construct a random directed polymer measure. Flandoli, Issoglio and Russo [7] were the first to consider multidimensional singular diffusions, but they require more regularity than in the previous works on the one-dimensional case (they consider the "Young regime", i.e., the distributional drift has regularity better than $-1/2$). Zhang and Zhao [22] study the ergodicity and they derive heat-kernel estimates for singular diffusions in the Young regime. Cannizzaro and Chouk [5] use paracontrolled distributions to extend the approach of [6] to higher dimensions and the results of [7] to more singular drifts. They apply this to construct a random polymer measure that is closely related to the parabolic Anderson model.

In this paper we follow the approach of Cannizzaro and Chouk, although we restrict our attention to the more regular Young regime. This is crucial for our arguments.

As already mentioned, Zhang and Zhao [22] also prove heat-kernel estimates for SDEs with distributional drifts in the Young regime. More precisely, they prove that there exist $c, C \geq 1$ such that for all $t \in (0,T]$ and $x, y \in \mathbb{R}^d$

$$\tfrac{1}{C} p(\tfrac{t}{c}, x-y) \leq |\Gamma_t(x,y)| \leq C p(ct, x-y).$$

Moreover, they give an upper bound on the gradient of the transition kernel, $\nabla\Gamma_t$. Here, the constant C implicitly depends on T and $\|b\|_{\mathscr{C}^{-\alpha}}$.

If b is the gradient of a function that does not dependent on time, then there is classical heat-kernel estimates for Γ, see for example Stroock [20, Theorem 4.3.9]. In that theorem we have $b = \nabla U$ for a smooth and bounded function U, but the estimate only depends on $\max U - \min U$, so by an approximation argument it extends to continuous and bounded U. This result is uniform in time, but also here the dependence of the constants on $\max U - \min U$ is implicit.

In another work by the authors together with W. König [17], our heat-kernel estimates are applied to derive the asymptotic behavior of the total mass of the parabolic Anderson model. In that application it is crucial to understand how the constant grows with t and the norm of b. Therefore, we need our "quantitative version" of the heat-kernel estimates.

1.2 Notation and conventions

We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N}_{-1} = \{-1\} \cup \mathbb{N}_0$. For the whole paper, d is an element of $\mathbb{N}$ and will denote the dimension of the space. For families $(a_i)_{i\in\mathbb{I}}, (b_i)_{i\in\mathbb{I}}$ in $\mathbb{R}$ for an index set $\mathbb{I}$, we write $a_i \lesssim b_i$ to denote the existence of a $C > 0$ such that $a_i \le C b_i$ for all $i \in \mathbb{I}$. We write C_{b} for the space of continuous bounded functions and C_{b}^∞ for the space of C^∞ functions for which all their derivatives are bounded functions. We abbreviate function spaces and Besov spaces by omitting "$(\mathbb{R}^d)$" in the notation, for example we abbreviate $B^\beta_{p,q}(\mathbb{R}^d)$ to $B^\beta_{p,q}$. Moreover, we write $\mathscr{C}^\beta$ for $B^\beta_{\infty,\infty}$ and $\mathscr{C}^\beta_p$ for $B^\beta_{p,\infty}$. We write $u \varolessthan v$ for the paraproduct between u and v (with the low frequencies of u and the high frequencies of v), and $u \odot v$ for the resonance product; we adopt the notation from [19] and refer to [2] as background material.

In the rest of the paper $(\rho_i)_{i\in\mathbb{N}_{-1}}$ is a *dyadic partition of unity*, meaning that ρ_{-1} is supported in a ball around 0, ρ_0 is supported in an annulus, $\rho_i(x) = \rho_0(2^{-i}x)$ for $i \in \mathbb{N}_0$, $\sum_{i\in\mathbb{N}_{-1}} \rho_i = \mathbb{1}$, $\frac{1}{2} \le \sum_{i\in\mathbb{N}_{-1}} \rho_i^2 \le 1$ and $\operatorname{supp}\rho_i \cap \operatorname{supp}\rho_j = \emptyset$ if $|i-j| \ge 2$. For $i \in \mathbb{N}_{-1}$ we write Δ_i for the corresponding Littlewood-Payley blocks ($\mathscr{F}$ denotes the Fourier transform)

$$\Delta_i f = \rho_i(\mathrm{D}) f = \mathscr{F}^{-1}(\rho_i \mathscr{F}(f)) = \mathscr{F}^{-1}(\rho_i) * f.$$

Moreover, we define $\Delta_{\ge 0} f$ to be the sum of all the positive Littlewood-Payley blocks:

$$\Delta_{\ge 0} f = \sum_{i\in\mathbb{N}_0} \Delta_i f.$$

2 Diffusions with distributional drift and their heat-kernel estimates

Throughout this section we fix $T > 0$**.** Let $\alpha \in (0, \frac{1}{2})$. For $b \in C([0,T], B^{-\alpha}_{\infty,1}(\mathbb{R}^d, \mathbb{R}^d))$ we consider the stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}B_t. \tag{5}$$

For $t > 0$ let $\mathscr{L}_t$ be the operator

$$\mathscr{L}_t = \tfrac{1}{2}\Delta + b_t \cdot \nabla. \tag{6}$$

We consider the following Cauchy problem for $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ with terminal condition ϕ:

$$\begin{cases} \partial_t u + \mathscr{L}_t u = 0 & \text{on } [0,T) \times \mathbb{R}^d, \\ u(T,\cdot) = \phi & \text{on } \mathbb{R}^d. \end{cases} \tag{7}$$

The solution theory for the Cauchy problem will be given in Proposition 2.4. We write u^ϕ for the solution to (7). But let us first discuss how to interpret (5) in terms of a martingale problem.

Definition 2.1. We say that a stochastic process $X = (X_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathbb{P})$ is a *solution to the SDE* (5) *on* $[0,T]$ *with initial condition* $X_0 = x$ if it satisfies the martingale problem for $((\mathscr{L}_t)_{t\in(0,T]}, \delta_x)$, i.e., if $\mathbb{P}(X_0 = x) = 1$ and for all $f \in C([0,T], L^\infty(\mathbb{R}^d))$, all $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ and for $u = u^\phi$ being the solution to the Cauchy problem (7), the process

$$\Big(u(t, X_t) - \int_0^t f(s, X_s)\,\mathrm{d}s\Big)_{t\in[0,T]}$$

is a martingale.

The martingale problem has a unique solution:

Theorem 2.2. [5, Theorem 1.2] *Let $\alpha \in (0, \frac{1}{2})$. For all $x \in \mathbb{R}^d$ and $b \in C([0,T], \mathscr{C}^{-\alpha}(\mathbb{R}^d, \mathbb{R}^d))$ there exists a unique solution to the martingale problem for $((\mathscr{L}_t)_{t\in(0,T]}, \delta_x)$, in the sense that there is a unique probability measure $\mathbb{P}_x$ on $\Omega = C([0,T], \mathbb{R}^d)$ such that the coordinate process $X_t(\omega) = \omega(t)$ satisfies the martingale problem for $((\mathscr{L}_t)_{t\in(0,T]}, \delta_x)$. Moreover, X is a strong Markov process under $\mathbb{P}_x$ and the measure $\mathbb{P}_x$ depends (weakly) continuously on the drift b.*

Remark 2.3. The continuity of the solution $\mathbb{P}$ in terms of the drift is not mentioned in [5, Theorem 1.2], but it can be extracted from their proof.

Observe that Theorem 2.2 also implies that there exists a unique probability measure $\mathbb{P}_{s,x}$ on $C([s,T], \mathbb{R}^d)$ such that the coordinate process satisfies the martingale problem for $((\mathscr{L}_t)_{t\in(s,T]}, \delta_x)$. This can be obtained by applying Theorem 2.2 to a shift of the drift, as is mentioned in Remark 1.4.

Next, our aim is to show that X admits a transition density $\Gamma_{s,t}$ for $0 \le s < t \le T$ (Proposition 2.9), which means that for $\varphi \in C_c(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ and with $\mathbb{P}_{s,x}$ as in Remark 2.3

$$\mathbb{E}_{s,x}[\varphi(X_t)] = \int_{\mathbb{R}^d} \varphi(y)\Gamma_{s,t}(x,y)\,\mathrm{d}y. \tag{8}$$

We do this by showing that $\Gamma_{t,T}(x,y) = u^{\delta_y}(T-t,x)$ for the solution u^{δ_y} to (7) with terminal condition $u(T,\cdot) = \delta_y$.

In order to construct the solution u^{δ_y} we have to slightly extend the results of [5]. Indeed, in [5, Theorem 3.1 and 3.2] the well-posedness of the Cauchy problem is shown for $\phi \in \mathscr{C}^\beta$ with $\beta \in (1+\alpha, 2-\alpha)$, and δ_z is not in this space. The solution theory in [5] is formulated in terms of mild solutions: A *mild solution* of (7) is a fixed point u of Φ, i.e., $\Phi u = u$, where Φ is defined on $C([0,T], \mathscr{S}') \cap [\bigcup_{p\in[1,\infty]} C([0,T), \mathscr{C}_p^\beta(\mathbb{R}^d))]$ for $\beta > 1+\alpha$ by

$$(\Phi u)_s = P_{T-s}\phi - \int_s^T P_{r-s}(b_r \cdot \nabla u_r)\,\mathrm{d}r, \tag{9}$$

where $P_t\phi := p(t,\cdot) * \phi$ for $t > 0$ and $P_0\phi = \phi$ (that Φ is well-defined follows by 2.6). Let us check this by doing formal calculations. This means we treat b as a nice function (say in C_b^∞). The calculations are supported by Lemma A.2. If $\Phi u = u$, $F(r) = \int_0^s f(s-r,r)\,\mathrm{d}r$, $f(s,r) = P_s(b_r \cdot \nabla u_r)$, then

$$\begin{aligned}\partial_s u_s &= -\Delta P_{T-s}\phi - P_0(b_s \cdot \nabla u_s) + \int_s^T \Delta P_{r-s}(b_r \cdot \nabla u_r)\,\mathrm{d}r \\ &= -\Delta u_s - b \cdot \nabla u_s = -\mathscr{L}_s u,\end{aligned}$$

so that u is a solution to (7). In order to allow δ_y as a terminal condition, we will consider a different space that "allows a blowup as $t \uparrow T$". However, for notational elegance, we instead consider a space with "a blowup at 0" and mention that u is a fixed point of Φ if and only if v given by $v(t,\cdot) = u(T-t,\cdot)$ is a fixed point of Ψ, given by

$$(\Psi v)_s = P_s\phi + \int_0^s P_{s-r}(b_{T-r} \cdot \nabla v_r)\,\mathrm{d}r, \tag{10}$$

Sanity check, if $u = \Phi u$, then

$$\begin{aligned} v_s = u_{T-s} = (\Phi u)_{T-s} &= P_s\phi - \int_{T-s}^{T} P_{r-(T-s)}(b_r \cdot \nabla u_r)\,\mathrm{d}r \\ &= P_s\phi - \int_0^s P_{s-q}(b_{T-q} \cdot \nabla u_{T-q})\,\mathrm{d}q \\ &= P_s\phi - \int_0^s P_{s-r}(b_{T-r} \cdot \nabla v_r)\,\mathrm{d}r. \end{aligned}$$

so that we call v a *mild solution* of

$$\begin{cases} \partial_t v - \mathscr{L}_{T-t} v = 0 & \text{on } (0,T] \times \mathbb{R}^d, \\ v(0,\cdot) = \phi & \text{on } \mathbb{R}^d. \end{cases} \tag{11}$$

Indeed, let us suppose that u is a solution to (7) and $v(t,\cdot) = u(T-t,\cdot)$. Then $v(0,x) = u(T,x) = \phi$ and because $\nabla v(s,\cdot) = \nabla u(T-s,\cdot)$,

$$\partial_t v(t,x) = -\partial_t u(T-t,x) = \mathscr{L}_{T-t} u(T-t,x) = \mathscr{L}_{T-t} v(t,x).$$

We will show that Ψ has a fixed point in the following space (for suitable δ, β). For $\delta \geq 0$, $\beta \in \mathbb{R}$ and $t > 0$ we define

$$\begin{aligned} \|u\|_{M_t^\delta \mathscr{C}_p^\beta} &= \sup_{s \in (0,t]} s^\delta \|u_s\|_{\mathscr{C}_p^\beta}, \\ M_t^\delta \mathscr{C}_p^\beta &= \{u \in C((0,t], \mathscr{C}_p^\beta) : \|u\|_{M_t^\delta \mathscr{C}_p^\beta} < \infty\}. \end{aligned}$$

The following proposition is a slight extension of [5, Theorem 3.1 and 3.2].

Proposition 2.4. *Let $\alpha \in (0, \frac{1}{2})$, $p \in [1,\infty]$ and $\gamma > \alpha - 1$. For $\phi \in \mathscr{C}_p^\gamma$, $b \in C([0,T], B_{\infty,1}^{-\alpha})$, $\beta \in (1+\alpha, 2-\alpha)$ and $\varepsilon > 0$ the Cauchy problem (7) has a unique mild solution $u^{\phi,b}$ in $C([0,T], \mathscr{C}_p^{(\gamma-\varepsilon)\wedge\beta}) \cap C([0,T), \mathscr{C}_p^\beta)$ such that $u_b^\phi(t) \in \mathscr{C}^\beta$ for all $t \in [0,T)$. Moreover, for all $t > 0$ the map $\mathscr{C}_p^\gamma \times C([0,T], B_{\infty,1}^{-\alpha}) \to \mathscr{C}^\beta$ given by $(\phi, b) \mapsto u^{\phi,b}(t,\cdot)$ is locally Lipschitz.*

Another difference with [5] is that we consider $b \in C([0,T], B_{\infty,1}^{-\alpha})$ instead of $b \in C([0,T], \mathscr{C}^{-\alpha})$. Since $B_{\infty,p}^{-\alpha} \subset \mathscr{C}^{-\alpha} \subset B_{\infty,p}^{-\alpha-\varepsilon}$ (as continuous embeddings), this does not make much of a difference. But our heat-kernel estimates depend on the $B_{\infty,1}^{-\alpha}$-norm and for their derivation it is more convenient to work with $B_{\infty,1}^{-\alpha}$.

Before we prove Proposition 2.4 we present two auxiliary facts, Lemma 2.5 and 2.6.

We write B for the beta function (see e.g. [1, Section 1.1]), which is the function $B : (0,\infty)^2 \to (0,\infty)$ given by

$$B(\beta, \gamma) = \int_0^1 \theta^{\gamma-1}(1-\theta)^{\beta-1}\,\mathrm{d}\theta. \tag{12}$$

Lemma 2.5. *Let $p \in [1,\infty]$, $\kappa \geq 0$, $\delta \in [0,1)$, $\alpha,\gamma \in \mathbb{R}$ and $\beta \in [-\alpha, 2-\alpha)$.*

There exists a $C > 0$ such that for all $t \in (0,1]$,

$$\|s \mapsto P_s\phi\|_{M_t^{\frac{\kappa}{2}}\mathscr{C}_p^{\gamma+\kappa}} \leq C\|\phi\|_{\mathscr{C}_p^{\gamma}}, \qquad \|P_t\phi - \phi\|_{\mathscr{C}_p^{\gamma-2\delta}} \leq Ct^{\delta}\|\phi\|_{\mathscr{C}_p^{\gamma}}, \tag{13}$$

$$\left\| s \mapsto \int_0^s P_{s-r}w_r \,\mathrm{d}r \right\|_{M_t^{\delta}\mathscr{C}_p^{\beta}} \leq Ct^{1-\frac{\alpha+\beta}{2}}\|w\|_{M_t^{\delta}\mathscr{C}_p^{-\alpha}}. \tag{14}$$

Proof. In [12, Lemma A.7] it is proven (for $p = \infty$, but can be caried on mutatis mutandis for general $p \in [1,\infty]$ by applying [2, Lemma 2.2] with p instead of ∞ at the right place) that for all $\kappa \geq 0$ and $\gamma \in \mathbb{R}$ there exists a $C > 0$ such that for all $t \in (0,1]$

$$\|P_t\phi\|_{\mathscr{C}_p^{\gamma+\kappa}} \leq Ct^{-\frac{\kappa}{2}}\|\phi\|_{\mathscr{C}_p^{\gamma}}, \tag{15}$$

which implies the first bound in (13). The second bound in (14) follows by (15) as

$$\begin{aligned}\|P_t\phi - \phi\|_{\mathscr{C}_p^{\gamma-2\delta}} &= \| \int_0^t \partial_s P_s\phi \,\mathrm{d}s\|_{\mathscr{C}_p^{\gamma-2\delta}} \leq \int_0^t \|P_s\Delta\phi\|_{\mathscr{C}_p^{\gamma-2\delta}} \,\mathrm{d}s \\ &\lesssim \int_0^t s^{-\frac{2-2\delta}{2}} \,\mathrm{d}s \|\Delta\phi\|_{\mathscr{C}_p^{\gamma-2}} \lesssim t^{\delta}\|\phi\|_{\mathscr{C}_p^{\gamma}}.\end{aligned}$$

The bound in (14) is also proven in [12, Lemma A.9], we give the proof to be self-contained. By applying (15) (with $\gamma = -\alpha$ and $\kappa = \alpha + \beta$ which is positive by assumption) we obtain for $t \in (0,1]$ and $s \in [0,t]$

$$\begin{aligned}\|P_{t-s}w_s\|_{\mathscr{C}_p^{\beta}} &\lesssim (t-s)^{-\frac{\alpha+\beta}{2}} \|w_s\|_{\mathscr{C}_p^{-\alpha}} \\ &\lesssim (t-s)^{-\frac{\alpha+\beta}{2}} s^{-\delta}\|w\|_{M_T^{\delta}\mathscr{C}_p^{-\alpha}},\end{aligned}$$

and thus

$$\begin{aligned}\left\| \int_0^t P_{t-s}w_s \,\mathrm{d}s \right\|_{\mathscr{C}_p^{\beta}} &\lesssim \int_0^t (t-s)^{-\frac{\alpha+\beta}{2}} s^{-\delta} \,\mathrm{d}s \|w\|_{M_T^{\delta}\mathscr{C}_p^{-\alpha}} \\ &\lesssim t^{-\delta+1-\frac{\alpha+\beta}{2}} B\left(1 - \tfrac{\alpha+\beta}{2}, 1-\delta\right) \|w\|_{M_T^{\delta}\mathscr{C}_p^{-\alpha}}.\end{aligned} \tag{16}$$

Observe that $1 - \frac{\alpha+\beta}{2}, 1-\delta \in (0,\infty)$ by assumption. This proves (14). ☐

2.6. Let $\alpha > 0$ and let $\beta > 1 + \alpha$ and $\varepsilon > 0$ be such that $1 + \alpha + \varepsilon \leq \beta$. Then we have by Theorem A.1 together with Bernstein's inequality ([2, Lemma 2.1 or 2.78]):

$$\|a \cdot \nabla w\|_{B_{p,\infty}^{-\alpha}} \lesssim \|a\|_{B_{\infty,1}^{-\alpha}} \|\nabla w\|_{B_{p,\infty}^{\alpha+\varepsilon}} \lesssim \|a\|_{B_{\infty,1}^{-\alpha}} \|w\|_{B_{p,\infty}^{\beta}}.$$

Proof of Proposition 2.4. If $\gamma \geq \beta$, then the statement follows directly from [5, Theorem 3.2]. Therefore, we assume that $\gamma < \beta$ and it is sufficient to show that the statement holds for "t_0" instead of "T", where t_0 will be chosen small, as we can extend the solution to $[t_0, T]$ by [5, Theorem 3.2].

As mentioned before, it is sufficient to consider the fixed point problem for Ψ as in (10) instead of Φ. Let us write Ψ_t^{ϕ} for Ψ as in (10) but with "T" replaced by "t". We will show that there exists a t_0 such that

(a) $\Psi_{t_0}^{\phi}$ has a unique fixed point in $M_{t_0}^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}$,

(b) $\Psi_{t_0}^{\phi}$ has a unique fixed point in $C([0,t_0],\mathscr{C}_p^{\gamma-\varepsilon})$,

(c) $\Psi_{t_0}^{\phi}$ maps $C((0,t_0],\mathscr{C}_p^{\beta})$ and thus $M_{t_0}^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}$ into $C([0,t_0],\mathscr{C}_p^{\gamma-\varepsilon})$, so that the fixed point in $C([0,t_0],\mathscr{C}_p^{\gamma-\varepsilon})$ agrees with the fixed point in $M_{t_0}^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}$,

(d) for all $t>0$ the map $\mathscr{C}_p^{\gamma}\times C([0,T],B_{\infty,1}^{-\alpha})\to\mathscr{C}_p^{\beta}$ given by $(\phi,b)\mapsto u^{\phi,b}(t,\cdot)$ is locally Lipschitz,

(e) the fixed point v satisfies $v(t)\in\mathscr{C}^{\beta}$ for $t\in(0,t_0]$ and the continuity in (d) can be shown for $p=\infty$, by showing that we can "increase the integrability parameter p to ∞".

First, we assume that $\gamma>-\alpha$ and show (a)–(e). After that we show how one can treat $\gamma\in(\alpha-1,-\alpha]$ too.

(a) By combining the observation in 2.6 with Lemma 2.5 with $\kappa=\beta-\gamma>0$ and $\delta=\frac{\beta-\gamma}{2}\in(0,1)$, which is in $(0,1)$ as $\beta-\gamma<2-\alpha+\alpha$, for $t\in(0,1]$ we see that Ψ_t^{ϕ} maps $M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}$ to itself, as

$$
\begin{aligned}
\|\Psi_t^{\phi}v\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}} &\lesssim \|\phi\|_{\mathscr{C}_p^{\gamma}}+\|s\mapsto\int_0^s P_{s-r}(b_{t-r}\cdot\nabla v_r)\,\mathrm{d}r\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}}\\
&\lesssim \|\phi\|_{\mathscr{C}_p^{\gamma}}+t^{1-\frac{\alpha+\beta}{2}}\|s\mapsto b_{t-s}\cdot\nabla v_s\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{-\alpha}}\\
&\lesssim \|\phi\|_{\mathscr{C}_p^{\gamma}}+t^{1-\frac{\alpha+\beta}{2}}\|b\|_{C_1B_{\infty,1}^{-\alpha}}\|v\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}},
\end{aligned}
\tag{17}
$$

and, moreover

$$
\|\Psi_t^{\phi}v-\Psi_t^{\phi}\tilde{v}\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}}\lesssim t^{1-\frac{\alpha+\beta}{2}}\|b\|_{C_1B_{\infty,1}^{-\alpha}}\|v-\tilde{v}\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}}. \tag{18}
$$

So for sufficiently small t_0 the map $\Psi_{t_0}^{\phi}$ is a contraction on the Banach space $M_{t_0}^{\frac{\beta-\gamma}{2}}\mathscr{C}_p^{\beta}$ and it has a unique fixed point in that space.

(b) When t_0 is as above, then $\Psi_{t_0}^{\phi}$ has a unique fixed point in $C([0,t_0],\mathscr{C}_p^{\gamma-\varepsilon})$ which follows from the following estimates which follow similarly as the above ones (use that $\gamma\le\beta$ and (14) with $\beta=-\alpha$ and $\delta=0$)

$$
\begin{aligned}
\|\Psi_t^{\phi}v\|_{C([0,t],\mathscr{C}_p^{\gamma-\varepsilon})} &\lesssim \|\phi\|_{\mathscr{C}_p^{\gamma-\varepsilon}}+t\|b\|_{C_1B_{\infty,1}^{-\alpha}}\|v\|_{C([0,t],\mathscr{C}_p^{\gamma-\varepsilon})},\\
\|\Psi_t^{\phi}v-\Psi_t^{\phi}\tilde{v}\|_{C([0,t],\mathscr{C}_p^{\gamma-\varepsilon})} &= \|s\mapsto\int_0^s P_{s-r}(b\cdot\nabla(v_r-\tilde{v}_r))\,\mathrm{d}r\|_{C([0,t],\mathscr{C}_p^{\gamma-\varepsilon})}\\
&\lesssim t\|s\mapsto b\cdot\nabla(v_s-\tilde{v}_s)\|_{C([0,t],\mathscr{C}_p^{-\alpha})}\\
&\lesssim t\|b\|_{C_1B_{\infty,1}^{-\alpha}}\|v-\tilde{v}\|_{C([0,t],\mathscr{C}_p^{\gamma-\varepsilon})}.
\end{aligned}
$$

(c) That $\Psi_{t_0}^{\phi}$ maps $C((0,t_0],\mathscr{C}_p^{\beta})$ into $C([0,t_0],\mathscr{C}_p^{\gamma-\varepsilon})$, which means that $\Psi_{t_0}^{\phi}(w)(t)$ converges to ϕ in $\mathscr{C}_p^{\gamma-\varepsilon}$ as $t\downarrow 0$, we see from the second bound in (13) and the following estimate (by 2.6, which follows similarly to (17))

$$\|\Psi^{\phi}_{t_0}(w)(t)-\phi\|_{\mathscr{C}^{\gamma-\varepsilon}_p} \leq \|P_t\phi-\phi\|_{\mathscr{C}^{\gamma-\varepsilon}_p} + t^{1-\frac{\alpha+\beta}{2}}\|b\|_{C_1 B^{-\alpha}_{\infty,1}}\|v\|_{C((0,t_0],\mathscr{C}^{\beta}_p)}.$$

(d) Let us write $v^{b,\phi}$ for the solution of (11) (with $\mathscr{L}_t$ as in (6)). To see the continuity of the solution with respect to b and ϕ, let $b_1, b_2 \in C([0,t_0], B^{-\alpha}_{\infty,1})$ and $\phi_1,\phi_2 \in \mathscr{C}^{\gamma}_p$. Let $v_i = v^{b_i,\phi_i}$ for $i \in \{1,2\}$. By Lemma 2.5 and by 2.6 we have

$$\begin{aligned}\|v_1-v_2\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}^{\beta}_p} \lesssim \|\phi_1-\phi_2\|_{\mathscr{C}^{\gamma}_p} &+ t^{1-\frac{\alpha+\beta}{2}}\|b_1\|_{C_t B^{-\alpha}_{\infty,1}}\|v_1-v_2\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}^{\beta}_p}\\ &+ t^{1-\frac{\alpha+\beta}{2}}\|b_1-b_2\|_{C_t B^{-\alpha}_{\infty,1}}\|v_2\|_{M_t^{\frac{\beta-\gamma}{2}}\mathscr{C}^{\beta}_p}.\end{aligned}$$

Hence there exists a $\delta \in (0,t_0)$ (small enough) such that

$$\|v_1-v_2\|_{M_\delta^{\frac{\beta-\gamma}{2}}\mathscr{C}^{\beta}_p} \lesssim \|\phi_1-\phi_2\|_{\mathscr{C}^{\gamma}_p} + \|b_1-b_2\|_{C_{t_0} B^{-\alpha}_{\infty,1}}\|v_2\|_{M_{t_0}^{\frac{\beta-\gamma}{2}}\mathscr{C}^{\beta}_p}. \tag{19}$$

So for $t \in (0,\delta]$ we obtain the desired continuity. By an iteration argument we can obtain the continuity for all $t \in (0,t_0]$, as for example for $t \in (\delta, 2\delta]$ we have $v_i(t) = v^{b,v_i(\delta)}(t-\delta)$.

(e) It remains to show that we can increase the integrability from p to ∞, i.e., that $v_t \in \mathscr{C}^{\beta}$ for all $t>0$ and that also as an element of $\mathscr{C}^{\beta}$ the solution v_t for fixed $t>0$ depends continuously on b and ϕ. First we show that if $t>0$, then $v_s \in \mathscr{C}^{\beta}$ for all $s>t$. To simplify notation we only consider the most extreme case $p=1$, but the argument for general p is essentially the same. Let $n \in \mathbb{N}_0$ be such that

$$n(\beta-\gamma) < d, \qquad (n+1)(\beta-\gamma) \geq d.$$

Write $p_0 = 1$ and for $i \in \{1,\dots,n\}$

$$p_i = \frac{d}{d-i(\beta-\gamma)} \in (1,\infty).$$

Then $\beta - \frac{d}{p_n} \geq \gamma$ and $\beta - d(\frac{1}{p_{i-1}} - \frac{1}{p_i}) = \gamma$ for all $i \in \{1,\dots,n-1\}$, indeed

$$\frac{d}{p_n} = d-n(\beta-\gamma)\begin{cases} >0 \\ = d-(n+1)(\beta-\gamma)+\beta-\gamma \leq \beta-\gamma, \\ d(\frac{1}{p_i}-\frac{1}{p_{i-1}}) = d-i(\beta-\gamma)-(d-(i-1)(\beta-\gamma)) = \gamma-\beta, \end{cases}$$

hence the Besov embedding theorem [2, Proposition 2.71] gives $\mathscr{C}^{\beta}_{p_{i-1}} \subset \mathscr{C}^{\gamma}_{p_i}$ for all $i \in \{1,\dots,n-1\}$, and $\mathscr{C}^{\beta}_{p_n} \subset \mathscr{C}^{\gamma}$. We have $v_{\frac{t}{n}} \in \mathscr{C}^{\beta}_1 \subset \mathscr{C}^{\gamma}_{p_1}$. By considering the equation (11) with initial condition $v_{\frac{t}{n}}$ we obtain that v_s is in $\mathscr{C}^{\beta}_{p_1}$ for $s > \frac{t}{n}$, in particular $v_{\frac{2}{n}t} \in \mathscr{C}^{\gamma}_{p_2}$. Repeating the argument we obtain $v_{\frac{i}{n}t} \in \mathscr{C}^{\gamma}_{p_i}$ for all $i \in \{1,\dots,n-1\}$ and $v_t \in \mathscr{C}^{\gamma}$, so indeed $v_s \in \mathscr{C}^{\beta}$ for all $s>t$. As t was arbitrary, we have shown that $v_t \in \mathscr{C}^{\beta}$ for all $t>0$. As all the inclusions $\subset$

above are given by continuous embeddings, the continuity of the solution with respect to ϕ and b follows from the continuity shown in (d).

We are left to show that we can also treat $\gamma \in (\alpha - 1, -\alpha]$. Let γ be as such. We choose $\tilde{\beta} \in (\alpha + 1, 2 - \alpha)$ such that $\tilde{\beta} - \gamma < 2$. Then we have $\tilde{\beta} > \gamma$ and $\frac{\tilde{\beta}-\gamma}{2} \in (0, 1)$, so that the conditions of observation 2.6 and Lemma 2.5 are satisfied. Hence we obtain also (17) and (18) with "$\tilde{\beta}$" instead of "β". So then we find a $\tilde{t}_0 \in (0, t_0)$ such that $\Phi^{\phi}_{\tilde{t}_0}$ has a fixed point $\tilde{v}$ in $M^{\frac{\tilde{\beta}-\gamma}{2}}_{\tilde{t}_0} \mathscr{C}^{\tilde{\beta}}_p$. Let w be the fixed point of $\Phi^{\frac{\beta-\tilde{\beta}}{2}}_{t_0 - \tilde{t}_0}$, which exists by (a) because $\tilde{\beta} > -\alpha$. Then $v(t) := \tilde{v}(t)$ for $t \in (0, \tilde{t}_0]$ and $v(t) := w(t - t_0)$ for $t \in (\tilde{t}_0, t_0]$ is a fixed point of $\Phi^{\phi}_{t_0}$ such that $(\tilde{t}_0, t_0] \mapsto \mathscr{C}^{\beta}_p$, $t \mapsto v(t)$ is continuous. As $\tilde{t}_0$ can be taken arbitrarily small, we conclude that $v \in C((0, t_0], \mathscr{C}^{\beta}_p)$. Similarly, we can obtain the continuity of the solution by using (19) with "(β, γ)" replaced by "$(\beta, \tilde{\beta})$" and using (19) with "(β, γ)" replaced by "$(\tilde{\beta}, \gamma)$". □

2.7. A direct computation using that $\Delta_i \delta_z(x) = \mathscr{F}^{-1}(\rho(2^{-i}\cdot))(x - z) = 2^{id}\mathscr{F}^{-1}(\rho)(2^i(x - z))$ for $i \geq 0$ shows that the Dirac delta δ_z is in $\mathscr{C}_p^{-d(1-\frac{1}{p})}$ for all $p \in [1, \infty]$, so in particular $\delta_z \in \mathscr{C}^0_1$. Moreover, $\mathscr{F}^{-1}\rho_i$ is a Schwartz function for fixed $i \geq -1$, and therefore $\mathbb{R}^d \ni z \mapsto \Delta_i \delta_z \in L^p$ is continuous. This easily yields that for $\varepsilon > 0$ the map $\mathbb{R}^d \ni z \mapsto \delta_z \in \mathscr{C}_1^{-\varepsilon}$ is continuous. Indeed, we have $\Delta_i \delta_0 = \mathscr{F}^{-1}(\rho_i \hat{\delta}_0) = \mathscr{F}^{-1}(\rho_i) = \mathscr{F}^{-1}(\rho(2^{-i}\cdot)) = 2^{id}\mathscr{F}^{-1}(\rho)(2^i\cdot)$ and thus for $i \geq 0$

$$\begin{aligned}\|\Delta_i \delta_0\|_{L^p} &= 2^{id}\Big(\int \mathscr{F}^{-1}(\rho)(2^i x)^p \,\mathrm{d}x\Big)^{\frac{1}{p}} = 2^{id}\Big(2^{-\frac{di}{p}} \int \mathscr{F}^{-1}(\rho)(x)^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\ &= 2^{id(1-\frac{1}{p})}\|\mathscr{F}^{-1}(\rho)\|_{L^p}.\end{aligned}$$

Hence $\delta_0 \in B_{p,\infty}^{-d(1-\frac{1}{p})}$, similarly $\delta_z \in \mathscr{C}_p^{-d(1-\frac{1}{p})}$.

Corollary 2.8 (of Proposition 2.4)**.** *Let $\alpha \in (0, \frac{1}{2})$ and $b \in C([0, T], B^{-\alpha}_{\infty,1}(\mathbb{R}^d, \mathbb{R}^d))$.*

For $t \in (0, T]$ and $n \in \mathbb{N}$ let $b^{(n)}_t = \sum_{i=1}^n \Delta_i b_t \in C^\infty_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^d)$ and let $\Gamma_{t,T}(x, y) = u^{\delta_y, b}(t, x)$ and $\Gamma^{(n)}_{t,T}(x, y) = u^{\delta_y, b^{(n)}}(t, x)$ (notation as in Proposition 2.4). Then $\Gamma_{t,T}$ and $\Gamma^{(n)}_{t,T}$ are continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and we have for all $\mu \in \mathbb{N}^d_0$ with $|\mu| \leq 1$:

$$\sup_{x,y \in \mathbb{R}^d} |\partial^\mu_x [\Gamma_{t,T}(x, y) - \Gamma^{(n)}_{t,T}(x, y)]| \xrightarrow{n\to\infty} 0.$$

Proof. The continuity follows from Proposition 2.4.

Because $\|b^{(n)}_s\|_{B^\alpha_{\infty,1}} \lesssim \|b_s\|_{B^\alpha_{\infty,1}}$ and $\|b^{(n)}_s - b_s\|_{B^{-\alpha}_{\infty,1}} \to 0$ we obtain by a "3ε argument" that

$$\|b^{(n)} - b\|_{C_t B^{-\alpha}_{\infty,1}} \to 0$$

Indeed: Let $\varepsilon > 0$. Let $\delta > 0$ be such that $|s - r| < \delta$ implies $\|b_s - b_r\|_{B^{-\alpha}_{\infty,1}} < \varepsilon$. Let $S \subset [0, t]$ be a finite set such that for all $r \in [0, t]$ there is an $s \in S$ with $|r - s| < \delta$. Let $N \in \mathbb{N}$ be such that $\|b_s^{(n)} - b_s\|_{B^{-\alpha}_{\infty,1}} < \varepsilon$ for all $n \geq N$ and $s \in S$. Then for all $r \in [0,t]$ we have with s being such that $|s-r| < \delta$, for all $n \geq N$

$$\|b_r^{(n)} - b_r\|_{B^{-\alpha}_{\infty,1}} \leq \|b_r^{(n)} - b_s^{(n)}\|_{B^{-\alpha}_{\infty,1}} + \|b_s^{(n)} - b_s\|_{B^{-\alpha}_{\infty,1}} + \|b_s - b_r\|_{B^{-\alpha}_{\infty,1}} \leq (M+2)\varepsilon,$$

where $M > 0$ is such that $\|b_s^{(n)}\|_{B^{\alpha}_{\infty,1}} \leq M\|b_s\|_{B^{\alpha}_{\infty,1}}$ for all $n \in \mathbb{N}$ and $s \geq 0$. This follows by Youngs inequality as $\sum_{i=1}^n \Delta_i a = (\mathscr{F}^{-1}\chi)(2^{-n}\cdot) * a$ for some χ such that $\mathscr{F}^{-1}$ is integrable, and because $\|\mathscr{F}^{-1}\chi)(2^{-n}\cdot)\|_{L^1} = \|\mathscr{F}^{-1}\chi)\|_{L^1}$ for all $n \in \mathbb{N}$.

As moreover $\sup_{y\in\mathbb{R}^d} \|\delta_y\|_{B^0_{1,\infty}} \lesssim 1$, Proposition 2.4 yields

$$\sup_{y\in\mathbb{R}^d} \|\Gamma_t(\cdot,y) - \Gamma_{t,n}(\cdot,y)\|_{\mathscr{C}^\beta} \to 0,$$

for all $\beta < 2-\alpha$. □

Proposition 2.9. *Let $\alpha \in (0,\frac{1}{2})$ and $b \in C([0,T], B^{-\alpha}_{\infty,1}(\mathbb{R}^d,\mathbb{R}^d))$. For $t \in [0,T)$ let $\Gamma_{t,T}\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be defined by $\Gamma_{t,T}(x,y) = u^{\delta_y}(t,x)$. Let $\mathbb{P}_{t,x}$ be the unique probability measure on $C([t,T],\mathbb{R}^d)$ such that the coordinate process X is a solution to the SDE* (5) *on $[t,T]$ with initial condition $X_t = x$. Then $\Gamma_{t,T}(x,\cdot)$ is the density of X_T under $\mathbb{P}_{t,x}$, i.e., $\mathbb{E}_{t,x}[\phi(X_T)] = \int_{\mathbb{R}^d} \phi(y)\Gamma_{t,T}(x,y)\,\mathrm{d}y$ for all $\phi \in C_c(\mathbb{R}^d)$.*

Proof. For b with values in C_b^∞ this is classical, see for example [10, Theorem 6.5.4]. So let $b^{(n)}$ and $\Gamma_{t,T}^{(n)}$ be as in Corollary 2.8 and for $x \in \mathbb{R}^d$ let $\mathbb{P}_{t,x}^{(n)}$ be the unique probability measure on $C([t,T],\mathbb{R}^d)$ such that the coordinate process X is a solution to the martingale problem for $((\mathscr{L}_s^{(n)})_{s\in(t,T]},\delta_x)$, where $\mathscr{L}_s^{(n)} = \frac{1}{2}\Delta + b_{T-s}^{(n)} \cdot \nabla$. Using that $\mathbb{P}_{t,x}^{(n)}$ weakly converges to $\mathbb{P}_{t,x}$ (Theorem 2.2) and the uniform convergence in Corollary 2.8 we obtain for $\phi \in C_c(\mathbb{R}^d)$:

$$\mathbb{E}_{t,x}[\phi(X_T)] = \lim_{n\to\infty} \mathbb{E}_{t,x}^{(n)}[\phi(X_T)] = \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(y)\Gamma_{t,T}^{(n)}(x,y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \phi(y)\Gamma_{t,T}(x,y)\,\mathrm{d}y.$$

□

3 Heat-kernel upper bounds

Here we prove the upper bound (2) of the heat-kernel estimates. We follow the "parametrix" approach from Friedman's book [9] to prove the heat-kernel estimates presented in Theorem 1.1. This means that we write Γ_t as a series (see Lemma 3.3) and bound each term in that series to obtain a bound for the whole series and thus for Γ_t. Usually the point of the parametrix is to deal with non-constant diffusion coefficients, but the approach is still useful for us despite the fact that we deal with constant diffusion coefficients.

Because of Corollary 2.8 we can restrict our attention to b in $C([0,T], C_b^\infty(\mathbb{R}^d,\mathbb{R}^d))$ and then extend the bounds to b in $C([0,T], B^{-\alpha}_{\infty,1}(\mathbb{R}^d,\mathbb{R}^d))$ by a limiting argument.

For the rest of this section we fix $\alpha \in (0,\frac{1}{2})$, and $c > 1$ as in Theorem 1.1 and $b \in C([0,\infty), C_b^\infty(\mathbb{R}^d,\mathbb{R}^d))$. (Instead of $[0,T]$ we consider $[0,\infty)$ for notational convenience.) Observe that $C_b^\infty \subset B^{-\alpha}_{\infty,1}$.

3.1. Let $g \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ and $a \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Let $(\tilde{\rho}_i)_{i \in \mathbb{N}_{-1}}$ be another dyadic partition of unity, but such that $\operatorname{supp} \tilde{\rho}_{-1} \cap \operatorname{supp} \rho_i = \emptyset$ for $i \in \mathbb{N}_0$ so that

$$\begin{aligned}\int_{\mathbb{R}^d} (\Delta_i a)(z)(\tilde{\Delta}_{-1} g)(z)\,\mathrm{d}z &= \int_{\mathbb{R}^d} \mathscr{F}^{-1}(\rho_i \hat{a})(z)\mathscr{F}^{-1}(\tilde{\rho}_{-1}\hat{g})(z)\,\mathrm{d}z \\ &= \int_{\mathbb{R}^d} \hat{a}(-z)\rho_i(z)\tilde{\rho}_{-1}(z)\hat{g}(z)\,\mathrm{d}z = 0,\end{aligned}$$

and thus

$$\int_{\mathbb{R}^d} (\Delta_{\geq 0} a)(z) g(z)\,\mathrm{d}z = \int_{\mathbb{R}^d} (\Delta_{\geq 0} a)(z)(\tilde{\Delta}_{\geq 0} g)(z)\,\mathrm{d}z.$$

By duality and Bernstein's inequality, see [2, Proposition 2.76 and Lemma 2.1], we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} a(z)\cdot g(z)\,\mathrm{d}z\Big| &\leq \Big|\int_{\mathbb{R}^d} \Delta_{-1} a(z)\cdot g(z)\,\mathrm{d}z\Big| + \Big|\int_{\mathbb{R}^d} \Delta_{\geq 0} a(z)\cdot g(z)\,\mathrm{d}z\Big| \\ &\lesssim \|\Delta_{-1} a\|_{L^\infty}\|g\|_{L^1} + \|\Delta_{\geq 0} a\|_{B^{-\alpha}_{\infty,1}}\|\tilde{\Delta}_{\geq 0} g\|_{B^{\alpha}_{1,\infty}} \\ &\lesssim \|\Delta_{-1} a\|_{L^\infty}\|g\|_{L^1} + \|\Delta_{\geq 0} a\|_{B^{-\alpha}_{\infty,1}}\left(\sup_{j\geq 0}\left\{\|\tilde{\Delta}_j g\|_{L^1}^{1-\alpha}(2^j\|\tilde{\Delta}_j g\|_{L^1})^{\alpha}\right\}\right) \\ &\lesssim \|\Delta_{-1} a\|_{L^\infty}\|g\|_{L^1} + \|\Delta_{\geq 0} a\|_{B^{-\alpha}_{\infty,1}}\|g\|_{L^1}^{1-\alpha}\|\nabla g\|_{L^1}^{\alpha}. \end{aligned} \tag{20}$$

We will apply the above bound for functions g that are Gaussian, therefore we will need estimates for derivatives of Gaussian functions. So we recall the following bound:

3.2. Let $p(t,x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{2t}|x|^2}$ for $(t,x) \in (0,\infty)\times\mathbb{R}^d$ be the standard Gaussian kernel. For the space derivatives $\partial^\mu p$ we have the following estimate:

$$\forall \mu \in \mathbb{N}_0^d\ \exists C > 0\ \forall (t,x) \in (0,\infty)\times\mathbb{R}^d: \quad |\partial^\mu p(t,x)| \leq C t^{-\frac{|\mu|}{2}} p(ct,x), \tag{21}$$

Indeed, when μ has a 0 for the t derivatives, then $\partial^\mu = t^{-\frac{|\mu|}{2}} P_\mu(\frac{x}{\sqrt{t}})p(t,x)$ for some polynomial P_μ of order $\leq |\mu|$. This can be proved by induction, for the induction step note that $\partial_{x_i} P_\mu(\frac{x}{\sqrt{t}})$ equals the product of another polynomial with $t^{-\frac{1}{2}}$. Note that one can bound $P(a)e^{-a}$ by Ce^{-ca} for all $c < 1$, where C depends on c.

The proof of the upper bound (2) essentially follows by iterating the previous two observations. To carry out the argument we need the following result, which allows us to write Γ as an infinite series.

Lemma 3.3. *For $x, y \in \mathbb{R}^d$ and $s, t > 0$ with $s < t$ we define*

$$\Psi^{y,1}_{s,t}(x) = -b(t-s,x)\cdot\nabla p(s,x-y), \tag{22}$$

and for $k \geq 2$

$$\Psi^{y,k+1}_{s,t}(x) = -\int_0^s \int_{\mathbb{R}^d} b(t-s,x)\cdot\nabla p(s-r,x-z)\Psi^{y,k}_{r,t}(z)\,\mathrm{d}z\,\mathrm{d}r. \tag{23}$$

Then for all $t > 0$ and $k \in \mathbb{N}$ the map $s \mapsto \Psi^{y,k}_{s,t}$ is in $L^\infty((0,t], L^1(\mathbb{R}^d))$. Moreover, (with $\Gamma_{s,t}$ as in Proposition 2.9)

$$\Gamma_{s,t}(x,y) = p(t-s, x-y) + \sum_{k=1}^{\infty} \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r, x-z)\Psi^{y,k}_{r,t}(z)\, \mathrm{d}z\, \mathrm{d}r. \tag{24}$$

Proof. By (21) we know that $\|\Psi^{y,1}_{s,t}\|_{L^1(\mathbb{R}^d)} \lesssim t^{-\frac{1}{2}}$ and therefore $s \mapsto \Psi^{y,1}_{s,t}$ is in $L^1((0,t], L^1(\mathbb{R}^d))$. For $k=2$ we have (for the last inequality remember the definition of the beta function (12))

$$\begin{aligned} \|\Psi^{y,2}_{s,t}\|_{L^1(\mathbb{R}^d)} &\lesssim \int_0^{t-s} \|\nabla p(t-s-r,\cdot) * \Psi^{y,1}_{r,t}\|_{L^1(\mathbb{R}^d)}\, \mathrm{d}r \\ &\lesssim \int_0^{t-s} (t-s-r)^{-\frac{1}{2}} r^{-\frac{1}{2}}\, \mathrm{d}r = B(\tfrac{1}{2}, \tfrac{1}{2}) \lesssim 1. \end{aligned}$$

One can repeat this line of argument and obtain $\|\Psi^{y,k+1}_{s,t}\|_{L^1(\mathbb{R}^d)} \lesssim 1$ for $k \geq 2$, locally uniformly in s. It remains to show (24). As $\Gamma_{s,t}(x,y) = u^{\delta_y}(s,x)$ where u^{δ_y} being the fixed point of the map Φ as in (9) with $\phi = \delta_y$, that is, with $u = u^{\delta_y}$,

$$\begin{aligned} (\Phi u)_s &= P_{t-s}\delta_y - \int_s^t P_{q-s}(b_q \cdot \nabla u_q)\, \mathrm{d}q \\ &= P_{t-s}\delta_y - \int_0^{t-s} P_{t-s-r}(b_{t-r} \cdot \nabla u_{t-r})\, \mathrm{d}r. \end{aligned}$$

From a Picard iteration it follows that Γ is the limit of the sequence $\Gamma^0_t = 0$,

$$\begin{aligned} &\Gamma^{k+1}_{s,t}(x,y) \\ &= p(t-s, x-y) - \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r, x-z)(b(t-r,z) \cdot \nabla_z \Gamma^k_{t-r,t}(z,y))\, \mathrm{d}z\, \mathrm{d}r. \end{aligned}$$

Therefore, $\Gamma^1_{s,t}(x,y) = p(t-s, x-y)$ and we obtain recursively

$$\begin{aligned} \Gamma^2_{s,t}(x,y) &= p(t-s,x-y) - \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r, x-z) b(t-r,z) \cdot \nabla p(r, z-y)\, \mathrm{d}z\, \mathrm{d}r \\ &= p(t-s,x-y) + \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r, x-z)\Psi^{y,1}_{r,t}(z)\, \mathrm{d}z\, \mathrm{d}r \\ \Gamma^3_{s,t}(x,y) &= p(t-s,x-y) + \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r, x-z)\Psi^{y,1}_{r,t}(z)\, \mathrm{d}z\, \mathrm{d}r \\ &\quad - \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r,x-z) b(t-r,z) \cdot \nabla \left(\int_0^r \int_{\mathbb{R}^d} p(r-q, z-w)\Psi^{y,1}_{q,t}(w)\, \mathrm{d}w\, \mathrm{d}q \right) \mathrm{d}z\, \mathrm{d}r \\ &= p(t-s,x-y) + \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r, x-z)\Psi^{y,1}_{r,t}(z)\, \mathrm{d}z\, \mathrm{d}r \\ &\quad + \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r, x-z)\Psi^{y,2}_{r,t}(z)\, \mathrm{d}z\, \mathrm{d}r \end{aligned}$$

(see also [9, Chapter 1.4])

$$\Gamma^{k+1}_{s,t}(x,y) = p(t-s,x-y) + \sum_{\ell=1}^{k} \int_0^{t-s} \int_{\mathbb{R}^d} p(t-s-r,x-z)\Psi^{y,\ell}_{r,t}(z)\,\mathrm{d}z\,\mathrm{d}r.$$

This proves (24). □

3.4. Now let us get back to Remark 1.4. Observe that in the right-hand side in (24) the dependence on t is in the $\Psi^{y,k}$ functions, and we see that the rest is a function of $t-s$. This allows us to take the first time variable, s, equal to zero, and proof the heat-kernel bounds as in Theorem 1.1. From now on we write "Γ_t" for "$\Gamma_{0,t}$".

Note that the first term appearing in the right-hand side of (24) is already bounded by the right-hand side of (2). Therefore, we will recursively estimate

$$\int_0^t \int_{\mathbb{R}^d} p(t-s,x-z)\Psi^{y,k}_{s,t}(z)\,\mathrm{d}z\,\mathrm{d}s.$$

This will be done with the help of some auxiliary lemmas, which follow below.

3.5. Let $\mu \in \mathbb{N}_0^d$, $t>0$, $k\in\mathbb{N}$, $x,y\in\mathbb{R}^d$ and $g\in L^1(\mathbb{R}^d)$. As we write $P_t g = p(t,\cdot)*g$ (see (9)), we have $\partial^\mu P_t g = \partial^\mu p(t,\cdot)*g$.

For any given norm $\|\cdot\|$ we will write $\|\nabla f\| = \sum_{i=1}^d \|\partial_i f\|$ and $\|\nabla^2 f\| = \sum_{i,j=1}^d \|\partial_{ij} f\|$.

Lemma 3.6. *There exists a $C>0$ (independent of b) such that for all $\mu\in\mathbb{N}_0^d$ with $|\mu|\le 2$, $y\in\mathbb{R}^d$ and $t,s,r\in(0,\infty)$ with $t>s>r$ and all $f\in L^1(\mathbb{R}^d)$, with $g_{t,s,r}(z) = b(t-r,z)\cdot\int_{\mathbb{R}^d}\nabla p(s-r,z-w)f(w)\,\mathrm{d}w$*

$$|\partial^\mu P_{t-s} g_{t,s,r}(x)| \le C(t-s)^{-\frac{|\mu|}{2}} p(ct,x-y)\Big(\|\Delta_{-1} b_{t-r}\|_{L^\infty}\left\|\tfrac{\nabla P_{s-r} f}{p(cs,\cdot-y)}\right\|_{L^\infty}$$
$$+\|\Delta_{\ge 0} b_{t-r}\|_{B^{-\alpha}_{\infty,1}}\Big[(t-s)^{-\frac{\alpha}{2}}\left\|\tfrac{\nabla P_{s-r} f}{p(cs,\cdot-y)}\right\|_{L^\infty} + \left\|\tfrac{\nabla P_{s-r} f}{p(cs,\cdot-y)}\right\|_{L^\infty}^{1-\alpha}\left\|\tfrac{\nabla^2 P_{s-r} f}{p(cs,\cdot-y)}\right\|_{L^\infty}^{\alpha}\Big]\Big). \quad (25)$$

Proof. We abbreviate $g_{t,s,r}$ by g. Observe that $g(z) = b(t-r,z)\cdot\nabla P_{s-r}f(z)$. Then, with $h:\mathbb{R}^d\to\mathbb{R}^d$, $h(z) = \partial^\mu p(t-s,x-z)\nabla P_{s-r}f(z)$, by (20)

$$\begin{aligned}|\partial^\mu P_{t-s} g(x)| &= \left|\int_{\mathbb{R}^d}\partial^\mu p(t-s,x-z) b(t-r,z)\cdot\nabla P_{s-r}f(z)\,\mathrm{d}z\right| \\ &\lesssim \|\Delta_{-1}b_{t-r}\|_{L^\infty}\|h\|_{L^1} + \|\Delta_{\ge 0} b_{t-r}\|_{B^{-\alpha}_{\infty,1}}\|h\|_{L^1}^{1-\alpha}\|\nabla h\|_{L^1}^{\alpha}.\end{aligned}$$

We estimate both $\|h\|_{L^1}$ and $\|\nabla h\|_{L^1}$. We use (21) and $\int_{\mathbb{R}^d} p(c(t-s),x-z)p(cs,z-y)\,\mathrm{d}z = p(c(t-s),\cdot)*p(cs,\cdot)(x-y) = p(ct,x-y)$ to obtain

$$\begin{aligned}\|h\|_{L^1} &= \int_{\mathbb{R}^d} |\partial^\mu p(t-s,x-z)\nabla P_{s-r}f(z)|\,\mathrm{d}z \\ &\lesssim \int_{\mathbb{R}^d}(t-s)^{-\frac{|\mu|}{2}} p(c(t-s),x-z)p(cs,z-y)\left\|\frac{\nabla P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}\mathrm{d}z \\ &= (t-s)^{-\frac{|\mu|}{2}} p(ct,x-y)\left\|\frac{\nabla P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}.\end{aligned}$$

Similarly, in combination with Leibniz's rule, we obtain

$$\begin{aligned}
&\|\nabla h\|_{L^1} = \left\|\nabla\big(\partial^\mu p(t-s,x-\cdot)\nabla P_{s-r}f\big)\right\|_{L^1} \\
&\le \left\|\partial^\mu \nabla p(t-s,x-\cdot)\nabla P_{s-r}f\right\|_{L^1} + \left\|\partial^\mu p(t-s,x-\cdot)\nabla^2 P_r f\right\|_{L^1} \\
&\lesssim (t-s)^{-\frac{|\mu|}{2}} p(ct,x-y)\left[(t-s)^{-\frac{1}{2}}\left\|\frac{\nabla P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty} + \left\|\frac{\nabla^2 P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}\right].
\end{aligned}$$

Using the above and that $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a,b \ge 0$ (indeed, by dividing by b^α we may instead assume $b = 1$; then, by a simple calculation we see that $(x+1)^\alpha \le x^\alpha + 1$: $\partial_x x^\alpha + 1 - (x+1)^\alpha = \alpha(x^{\alpha-1} - (x+1)^{\alpha-1}) \ge 0$ for $x \ge 0$) we obtain (25).

$$\begin{aligned}
\|h\|_{L^1}^{1-\alpha}\|\nabla h\|_{L^1}^{\alpha} &\lesssim (t-s)^{-\frac{|\mu|}{2}} p(ct,x-y)\left\|\frac{\nabla P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}^{1-\alpha} \\
&\quad\times\left[(t-s)^{-\frac{1}{2}}\left\|\frac{\nabla P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty} + \left\|\frac{\nabla^2 P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}\right]^\alpha \\
&\lesssim (t-s)^{-\frac{|\mu|}{2}} p(ct,x-y)\left\|\frac{\nabla P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}^{1-\alpha} \\
&\quad\times\left[(t-s)^{-\frac{\alpha}{2}}\left\|\frac{\nabla P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}^\alpha + \left\|\frac{\nabla^2 P_{s-r}f}{p(cs,\cdot-y)}\right\|_{L^\infty}^\alpha\right].
\end{aligned}$$

□

3.7. Now we apply the above lemma to our setting. But first, let us introduce some notation. For $k \in \mathbb{N}$, $t \ge 0$, $i \in \{0,1\}$, and $\beta \in \{0,\alpha\}$ we write

$$\mathscr{I}_{i,k}^\beta(t) = \sup_{y\in\mathbb{R}^d}\int_0^t \left\|\frac{\nabla^i P_{t-s}[\Psi_{s,t}^{y,k}]}{p(ct,\cdot-y)}\right\|_{L^\infty}^{1-\beta}\left\|\frac{\nabla^{i+1} P_{t-s}[\Psi_{s,t}^{y,k}]}{p(ct,\cdot-y)}\right\|_{L^\infty}^{\beta}\mathrm{d}s.$$

We are interested in the bounds for $\mathscr{I}_{i,k}^0$ only. But in order to describe a recursive relation for them, as we will see in the next lemma, we also need the $\mathscr{I}_{i,k}^\alpha$'s.

Lemma 3.8. *Let $C > 0$ be as in Lemma 3.6. For all $k \in \mathbb{N}$, $t \ge 0$, $i \in \{0,1\}$ and $\beta \in \{0,\alpha\}$*

$$\begin{aligned}
\mathscr{I}_{i,k+1}^\beta(t) \le C\int_0^t (t-s)^{-\frac{i+\beta}{2}}\Big(&\|\Delta_{-1}b\|_{C_tL^\infty}\mathscr{I}_{1,k}^0(s) \\
&+ \|\Delta_{\ge 0}b\|_{C_tB_{\infty,1}^{-\alpha}}\big[(t-s)^{-\frac{\alpha}{2}}\mathscr{I}_{1,k}^0(s) + \mathscr{I}_{1,k}^\alpha(s)\big]\Big)\,\mathrm{d}s. \qquad (26)
\end{aligned}$$

Proof. We claim that the following holds. For all $k \in \mathbb{N}$, $y \in \mathbb{R}^d$ and $i \in \{0,1,2\}$

$$\begin{aligned}
\left\|\frac{\nabla^i P_{t-s}[\Psi_{s,t}^{y,k+1}]}{p(cs,\cdot-y)}\right\|_{L^\infty} &\le C(t-s)^{-\frac{i}{2}}\Bigg(\|\Delta_{-1}b\|_{C_tL^\infty}\int_0^s\left\|\frac{\nabla P_{s-r}[\Psi_{r,t}^{y,k}]}{p(cs,\cdot-y)}\right\|_{L^\infty}\mathrm{d}r \\
&+ \|\Delta_{\ge 0}b\|_{C_tB_{\infty,1}^{-\alpha}}\Bigg[(t-s)^{-\frac{\alpha}{2}}\int_0^s\left\|\frac{\nabla P_{s-r}[\Psi_{r,t}^{y,k}]}{p(cs,\cdot-y)}\right\|_{L^\infty}\mathrm{d}r \\
&+ \int_0^s\left\|\frac{\nabla P_{s-r}[\Psi_{r,t}^{y,k}]}{p(cs,\cdot-y)}\right\|_{L^\infty}^{1-\alpha}\left\|\frac{\nabla^2 P_{s-r}[\Psi_{r,t}^{y,k}]}{p(cs,\cdot-y)}\right\|_{L^\infty}^{\alpha}\mathrm{d}r\Bigg]\Bigg). \qquad (27)
\end{aligned}$$

From this (26) follows by definition of $\mathscr{I}_k^\beta$. Now let us prove (27). Let $g_{t,s,r}$ be as in Lemma 3.6 with $f = \Psi_{r,t}^{y,k}$. Observe that by definition of $\Psi_{s,t}^{y,k+1}$ (23) we can write

$$\Psi_{s,t}^{y,k+1}(z) = \int_0^s b(t-r,z) \cdot \nabla P_{s-r}[\Psi_{r,t}^{y,k}](z)\,\mathrm{d}r = \int_0^s g_{t,s,r}(z)\,\mathrm{d}r,$$

so that (one can verify the interchange of integrals by Fubini's theorem and using Lemma 3.3)

$$|\nabla^i P_{t-s}[\Psi_{s,t}^{y,k+1}](x)| \leq \int_0^s |\nabla^i P_{t-s} g_{t,s,r}(x)|\,\mathrm{d}r.$$

With this, (27) follows from (25). □

In the proof of Lemma 3.10 we will use the following bound for the beta function (see (12)).

Lemma 3.9. *Let $\delta \in (0,1]$. Then $M_\delta := \sup\{B(\beta,\gamma)\gamma^\beta : (\beta,\gamma) \in [\delta,1] \times [\delta,\infty)\} < \infty$. Hence, for all $(\beta,\gamma) \in [\delta,1] \times [\delta,\infty)$,*

$$B(\beta,\gamma) = B(\gamma,\beta) \leq M_\delta \gamma^{-\beta}.$$

Proof. By [1, Theorem 1.1.4 and Theorem 1.4.1] we have for $\gamma,\beta > 0$

$$B(\beta,\gamma) = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}, \quad \text{and} \quad \lim_{\gamma\to\infty} \frac{\Gamma(\gamma)}{\sqrt{2\pi}\gamma^{\gamma-\frac{1}{2}}e^{-\gamma}} = 1.$$

From this we deduce the following. Let $\beta_n \to \beta$ for some $\beta \in [\delta,1]$ and $\gamma_n \to \infty$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{B(\beta_n,\gamma_n)\gamma_n^{\beta_n}}{\Gamma(\beta_n)} &= \lim_{n\to\infty} \frac{\sqrt{2\pi}\gamma_n^{\gamma_n-\frac{1}{2}}e^{-\gamma_n}\gamma_n^{\beta_n}}{\sqrt{2\pi}(\gamma_n+\beta_n)^{\gamma_n+\beta_n-\frac{1}{2}}e^{-(\gamma_n+\beta_n)}} \\
&= \lim_{n\to\infty} (1+\frac{\beta_n}{\gamma_n})^{-(\gamma_n+\beta_n-\frac{1}{2})}e^{\beta_n} \\
&= \lim_{\gamma\to\infty} (1+\frac{\beta_n}{\gamma_n})^{-\gamma_n}e^{\beta_n} = e^{-\beta_n}e^{\beta_n} = 1.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} B(\beta_n,\gamma_n)\gamma_n^{\beta_n} = \Gamma(\beta),$$

so that from the continuity of Γ it follows that $(\beta,\gamma) \mapsto B(\beta,\gamma)\gamma^\beta$ is a bounded function on $[\delta,1] \times [\delta,\infty)$. □

Let us now use the recursive relation for $\mathscr{I}_{i,k}^\beta$ and the bounds on the beta function to obtain estimates for $\mathscr{I}_{i,k}^\beta$:

Lemma 3.10. *Let $C > 0$ be as in Lemma 3.6 and let $M = 8M_{\frac{1}{2}-\alpha}$ with M_δ as in Lemma 3.9. There exists a $K > 0$ (independent of b) such that for all $k \in \mathbb{N}$, $t > 0$, $\beta \in \{0,\alpha\}$ and $i \in \{0,1\}$*

$$\mathscr{I}_{i,k}^\beta(t) \leq K \sum_{\substack{m,n\in\mathbb{N}_0:\\ m+n=k}} t^{-\frac{i+\beta}{2}} \frac{(CM\|\Delta_{-1}b\|_{C_tL^\infty}t^{\frac{1}{2}})^m}{(m!)^{\frac{1-\beta}{2}}} \frac{(CM\|\Delta_{\geq 0}b\|_{C_tB_{\infty,1}^{-\alpha}}t^{\frac{1-\alpha}{2}})^n}{(n!)^{\frac{1-\alpha-\beta}{2}}}. \tag{28}$$

Proof. We give a proof by induction. Instead of "$\|\Delta_{-1}b\|_{C_tL^\infty}$" and "$\|\Delta_{\geq 0}b\|_{C_tB^{-\alpha}_{\infty,1}}$" we will write "$X$" and "$Y$", respectively.

• The induction start, $k = 1$:

We have for $\mu \in \mathbb{N}_0^d$ with $|\mu| \leq 2$

$$\partial^\mu P_{t-s}[\Psi^{y,1}_{s,t}](x) = \int_{\mathbb{R}^d} \partial^\mu p(t-s, x-z)\Psi^{y,1}_{s,t}(z)\,\mathrm{d}z = \int_{\mathbb{R}^d} b(z)\cdot g_\mu(z)\,\mathrm{d}z$$

with $g_\mu(z) = \nabla p(s, z-y)\partial^\mu p(t-s, x-z)$. By (21) there exists a $K > 0$ such that for all $\mu, \nu \in \mathbb{N}_0^d$ with $|\mu| \leq 2$ and $|\nu| \leq 1$:

$$|g_\mu(z)| \leq K(t-s)^{-\frac{|\mu|}{2}} s^{-\frac{1}{2}} p(cs, z-y)p(c(t-s), x-z),$$
$$|\partial^\nu g_\mu(z)| \leq K(t-s)^{-\frac{|\mu|}{2}} s^{-\frac{1}{2}}[(t-s)^{-\frac{1}{2}} + s^{-\frac{1}{2}}]p(cs, z-y)p(c(t-s), x-z).$$

Therefore, by (20), for $j \in \{0, 1, 2\}$

$$\left|\nabla^j P_{t-s}[\Psi^{y,k}_{s,t}](x)\right| \leq K(t-s)^{-\frac{j}{2}} s^{-\frac{1}{2}}\Big(X + Y[(t-s)^{-\frac{\alpha}{2}} + s^{-\frac{\alpha}{2}}]\Big)p(ct, x-y),$$

and thus

$$\left\|\frac{\nabla^j P_{t-s}[\Psi^{y,1}_{s,t}]}{p(ct, \cdot - y)}\right\|_{L^\infty} \leq K(t-s)^{-\frac{j}{2}} s^{-\frac{1}{2}}\Big(X + Y[(t-s)^{-\frac{\alpha}{2}} + s^{-\frac{\alpha}{2}}]\Big),$$

so that for $i \in \{0, 1\}$

$$\begin{aligned}\mathscr{I}^\beta_{i,1}(t) &\leq K\int_0^t (t-s)^{-\frac{i+\beta}{2}} s^{-\frac{1}{2}}\Big(X + Y[(t-s)^{-\frac{\alpha}{2}} + s^{-\frac{\alpha}{2}}]\Big)\,\mathrm{d}s \\ &\leq t^{-\frac{i+\beta}{2}} K\Big(B(\tfrac{2-i-\beta}{2}, \tfrac{1}{2})Xt^{\frac{1}{2}} + \big[B(\tfrac{2-i-\alpha-\beta}{2}, \tfrac{1}{2}) + B(\tfrac{2-i-\beta}{2}, \tfrac{1-\alpha}{2})\big]Yt^{\frac{1-\alpha}{2}}\Big).\end{aligned}$$

Hence, for $k = 1$, the inequality (28) follows by applying Lemma 3.9 for the beta functions and using that $\delta \mapsto M_\delta$ is decreasing:

$$\begin{aligned}B(\tfrac{2-i-\beta}{2}, \tfrac{1}{2}) &\leq M_{\frac{2-i-\beta}{2}}(\frac{1}{2})^{-\frac{2-i-\beta}{2}} \leq 2M_{\frac{1}{2}-\alpha} \leq M, \\ B(\tfrac{2-i-\alpha-\beta}{2}, \tfrac{1}{2}) &\leq M_{\frac{2-i-\alpha-\beta}{2}} 2^{\frac{1-\alpha-\beta}{2}} \leq M, \\ B(\tfrac{2-i-\beta}{2}, \tfrac{1-\alpha}{2}) &\leq M_{\frac{2-i-\beta}{2}}(\tfrac{1-\alpha}{2})^{-\frac{1-\beta}{2}} \leq M_{\frac{1}{2}-\alpha}4^{\frac{1-\beta}{2}} \leq M.\end{aligned}$$

• The induction step, from k to $k+1$:

Let $k \in \mathbb{N}$ and assume that (28) holds. Then by Lemma 3.8

$$\begin{aligned}\mathscr{I}^\beta_{i,k+1}(t) &\leq C\int_0^t (t-s)^{-\frac{i+\beta}{2}}\Big(X\mathscr{I}^0_{1,k}(s) + Y[(t-s)^{-\frac{\alpha}{2}}\mathscr{I}^0_{1,k}(s) + \mathscr{I}^\alpha_{1,k}(s)]\Big)\,\mathrm{d}s \\ &\leq KC\sum_{\substack{m,n\in\mathbb{N}_0:\\ m+n=k}} \frac{(CMX)^m}{(m!)^{\frac{1-\beta}{2}}}\frac{(CMY)^n}{(n!)^{\frac{1-\alpha-\beta}{2}}} \\ &\qquad \times \int_0^t (t-s)^{-\frac{i+\beta}{2}} s^{-\frac{1}{2}+\frac{m}{2}+n\frac{1-\alpha}{2}}\Big(X + Y[(t-s)^{-\frac{\alpha}{2}} + s^{-\frac{\alpha}{2}}]\Big)\,\mathrm{d}s.\end{aligned}$$

We bound the latter integral, for which we have the following identity:

$$\int_0^t (t-s)^{-\frac{i+\beta}{2}} s^{-\frac{1}{2}+\frac{m}{2}+n\frac{1-\alpha}{2}} \left(X + Y[(t-s)^{-\frac{\alpha}{2}} + s^{-\frac{\alpha}{2}}]\right) \mathrm{d}s$$
$$= t^{-\frac{i+\beta}{2}} t^{\frac{m}{2}+n\frac{1-\alpha}{2}} \Big(X t^{\frac{1}{2}} B(\tfrac{1-\beta}{2}, \tfrac{m+1+n(1-\alpha)}{2})$$
$$+ Y t^{\frac{1-\alpha}{2}} \left[B(\tfrac{1-\alpha-\beta}{2}, \tfrac{m+1+n(1-\alpha)}{2}) + B(\tfrac{1-\beta}{2}, \tfrac{m+(n+1)(1-\alpha)}{2}) \right] \Big).$$

This shows that the power of t is the right one. We bound the beta function terms to finish the proof. By Lemma 3.9 we have

$$B(\tfrac{1-\beta}{2}, \tfrac{m+1+n(1-\alpha)}{2}) \leq M_{\frac{1-\beta}{2}} \left(\tfrac{m+1+n(1-\alpha)}{2}\right)^{-\frac{1-\beta}{2}} \leq 4M_{\frac{1}{2}-\alpha} (m+1)^{-\frac{1-\beta}{2}},$$
$$B(\tfrac{1-\alpha-\beta}{2}, \tfrac{m+1+n(1-\alpha)}{2}) \leq M_{\frac{1-\alpha-\beta}{2}} \left(\tfrac{m+1+n(1-\alpha)}{2}\right)^{-\frac{1-\alpha-\beta}{2}} \leq 4M_{\frac{1}{2}-\alpha} (n+1)^{-\frac{1-\alpha-\beta}{2}},$$
$$B(\tfrac{1-\beta}{2}, \tfrac{m+(n+1)(1-\alpha)}{2}) \leq M_{\frac{1-\beta}{2}} \left(\tfrac{m+(n+1)(1-\alpha)}{2}\right)^{-\frac{1-\beta}{2}} \leq 4M_{\frac{1}{2}-\alpha} (n+1)^{-\frac{1-\alpha-\beta}{2}}.$$

Here we used that $\frac{1-\alpha}{2} \in (\frac{1}{4}, \frac{1}{2})$

$$\left(\frac{1-\alpha}{2}\right)^{-\frac{1-\alpha-\beta}{2}} \leq 4^{\frac{1-\alpha-\beta}{2}} \leq 4.$$

□

Remark 3.11. The restriction $\alpha \in (0, \frac{1}{2})$ in Lemma 3.10 is necessary since $M = 4M_{\frac{1}{2}-\alpha}$ diverges as $\alpha \uparrow \frac{1}{2}$ (see see the definition of M_δ in Lemma 3.9). This is not unexpected, since for $\alpha > \frac{1}{2}$ we are no longer in the Young regime and we would need techniques like paracontrolled distributions or regularity structures to solve the equation for Γ.

Lemma 3.10 together with the following basic inequality constitutes the proof of Theorem 1.1.

Lemma 3.12. *Let $\beta \in (0,1)$. Then there exists an $L > 0$ such that for $z \geq 0$*

$$\sum_{k=0}^{\infty} \frac{z^k}{(k!)^\beta} \leq L \exp(L z^{\frac{1}{\beta}}).$$

Proof. Let $\delta > 0$. By writing $z^k = ((1+\delta)z)^k (1+\delta)^{-k}$ we get with Hölder's inequality

$$\sum_{k=0}^{\infty} \frac{z^k}{(k!)^\beta} \leq \left(\sum_{k=0}^{\infty} \left(\frac{((1+\delta)z)^k}{(k!)^\beta} \right)^{\frac{1}{\beta}} \right)^{\beta} \left(\sum_{k=0}^{\infty} (1+\delta)^{-\frac{k}{1-\beta}} \right)^{1-\beta} \simeq \exp(\beta(1+\delta)^{\frac{1}{\beta}} z^{\frac{1}{\beta}}).$$

□

Lemma 3.13. *There exists a $C > 0$ (independent of b) such that for all $\mu \in \mathbb{N}_0^d$ with $|\mu| \le 1$, and for all $t > 0$, $x, y \in \mathbb{R}^d$,*

$$\partial_x^\mu \Gamma_t(x,y) = \partial_x^\mu p(t, x-y) + \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} \partial_x^\mu p(t-s, x-z)\Psi_{s,t}^{y,k}(z)\,\mathrm{d}z\,\mathrm{d}s, \tag{29}$$

$$\begin{aligned}&|\partial_x^\mu \Gamma_t(x,y) - \partial_x^\mu p(t,x-y)| \\ &\quad \le Ct^{-\frac{|\mu|}{2}} p(ct, x-y)(\|\Delta_{-1}b\|_{C_tL^\infty} t^{\frac{1}{2}} \vee \|\Delta_{\ge 0}b\|_{C_tB^{-\alpha}_{\infty,1}} t^{\frac{1-\alpha}{2}}) \\ &\qquad \times \exp\left(Ct\Big[\|\Delta_{-1}b\|^2_{C_tL^\infty} + \|\Delta_{\ge 0}b\|^{\frac{2}{1-\alpha}}_{C_tB^{-\alpha}_{\infty,1}}\Big]\right).\end{aligned} \tag{30}$$

Proof. To show both (29) and (30) it is sufficient to estimate the series with the modulus of each term in the series in the right-hand side of (29) by the right-hand side of (30).

Let K, C, M be as in Lemma 3.10. Again, we will write "X" and "Y" instead of "$\|\Delta_{-1}b\|_{C_tL^\infty}$" and "$\|\Delta_{\ge 0}b\|_{C_tB^{-\alpha}_{\infty,1}}$". With $i = |\mu|$

$$\begin{aligned}\sum_{k=1}^\infty \int_0^t \left|\int_{\mathbb{R}^d} \partial_x^\mu p(t-s, x-z)\Psi^{y,k}(s,z)\,\mathrm{d}z\right| \mathrm{d}s &\le \left(\sum_{k=1}^\infty \mathscr{I}^0_{i,k}(t)\right) p(ct, x-y) \\ &\le Kt^{-\frac{i}{2}} p(ct, x-y) \sum_{\substack{m,n\in\mathbb{N}_0:\\ m+n\ge 1}} \frac{(CMXt^{\frac{1}{2}})^m}{(m!)^{\frac{1}{2}}} \frac{(CMYt^{\frac{1-\alpha}{2}})^n}{(n!)^{\frac{1-\alpha}{2}}} \\ &\le Kt^{-\frac{i}{2}} p(ct, x-y) CM(Xt^{\frac{1}{2}} + Yt^{\frac{1-\alpha}{2}}) \\ &\qquad \times \left(\sum_{m\in\mathbb{N}_0} \frac{(CMXt^{\frac{1}{2}})^m}{(m!)^{\frac{1}{2}}}\right)\left(\sum_{n\in\mathbb{N}_0} \frac{(CMYt^{\frac{1-\alpha}{2}})^n}{(n!)^{\frac{1-\alpha}{2}}}\right).\end{aligned}$$

Indeed, for $a, b > 0$

$$\begin{aligned}\sum_{\substack{m,n\in\mathbb{N}_0:\\ m+n\ge 1}} \frac{a^m}{(m!)^{\frac{1}{2}}} \frac{b^n}{(n!)^{\frac{1-\alpha}{2}}} &\le \sum_{m,n\in\mathbb{N}_0} \frac{a^{m+1}}{((m+1)!)^{\frac{1}{2}}} \frac{b^n}{(n!)^{\frac{1-\alpha}{2}}} + \sum_{m,n\in\mathbb{N}_0} \frac{a^m}{(m!)^{\frac{1}{2}}} \frac{b^{n+1}}{((n+1)!)^{\frac{1-\alpha}{2}}} \\ &\le (a+b) \sum_{m,n\in\mathbb{N}_0} \frac{a^m}{(m!)^{\frac{1}{2}}} \frac{b^n}{(n!)^{\frac{1-\alpha}{2}}}.\end{aligned}$$

Now by applying Lemma 3.12 in both cases $z = CMXt^{\frac{1}{2}}$, $\beta = \frac{1}{2}$ and $z = CMYt^{\frac{1-\alpha}{2}}$, $\beta = \frac{1-\alpha}{2}$, we obtain the desired bound. □

Proof of the heat-kernel upper bound (2) *of Theorem 1.1.* This is a direct consequence of Lemma 3.13, as there exists a $K > 0$ such that for all $t \ge 0$

$$Ct(X \vee Yt^{-\frac{\alpha}{2}}) \le \exp\left(Kt[X^2 + Y^{\frac{2}{1-\alpha}}]\right).$$

□

4 Heat-kernel lower bounds

The lower bound follows from Lemma 3.13 together with the next result, which is a small variation of [20, Lemma 4.3.8].

Lemma 4.1. *Let* $q_t : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ *for all* $t \in [0,\infty)$. *Suppose that* $(q_t)_{t\in[0,\infty)}$ *satisfies the Chapman-Kolmogorov equations, i.e.,* $q_{t+s}(x,y) = \int_{\mathbb{R}^d} q_t(x,z)q_s(z,y)\,\mathrm{d}z$. *Let* $a, b > 0$. *Suppose that* $q_t(x,y) \geq bt^{-\frac{d}{2}}$ *for all* $t \in (0,a]$ *and* $x,y \in \mathbb{R}^d$ *with* $|x-y| \leq \sqrt{t}$. *Then there exist a* $\kappa \in (0,1)$ *and an* $M > 1$, *which only depends on* b *and* d, *such that for all* $t \in [0,\infty)$ *and* $x,y \in \mathbb{R}^d$

$$q_t(x,y) \geq M^{-1-\frac{t}{a}} p(\kappa t, x-y).$$

Proof. By following the first step of the proof of [20, Lemma 4.3.8] we find a $\kappa \in (0,1)$ and a $M > 1$ which depend only on b and d such that for all $t \in (0,a]$ and $x,y \in \mathbb{R}^d$

$$q_t(x,y) \geq M^{-1} p(\kappa t, x-y).$$

Let $t > a$ and $n = \lceil \frac{t}{a} \rceil$. Then for all $x,y \in R^d$

$$\begin{aligned} q_t(x,y) &= \int_{(\mathbb{R}^d)^{n-1}} q_{\frac{t}{n}}(x,z_1)q_{\frac{t}{n}}(z_1,z_2)\cdots q_{\frac{t}{n}}(z_{n-1},y)\,\mathrm{d}z \\ &\geq \int_{(\mathbb{R}^d)^{n-1}} M^{-n} p(\kappa\tfrac{t}{n}, x-z_1)p(\kappa\tfrac{t}{n}, z_1-z_2)\cdots p(\kappa\tfrac{t}{n}, z_{n-1}-y)\,\mathrm{d}z \\ &\geq M^{-1-\frac{t}{a}} p(\kappa t, x-y). \end{aligned}$$

□

Now we can prove the heat-kernel lower bounds:

Proof of the heat-kernel lower bound (3) *of Theorem 1.1.* We want to apply Lemma 4.1. Therefore we will find an a such that the condition is satisfied. Once more we will write "X" and "Y" instead of "$\|\Delta_{-1}b\|_{C_tL^\infty}$" and "$\|\Delta_{\geq 0}b\|_{C_tB^{-\alpha}_{\infty,1}}$". Let us also take $X = \|\Delta_{-1}b\|_{C_tL^\infty}$ and $Y = \|\Delta_{\geq 0}b\|_{C_tB^{-\alpha}_{\infty,1}}$. Let $\alpha \in (0,\frac{1}{2})$, $c > 1$ and $C > 0$ be as in Lemma 3.13. Then (30) gives for $a > 0$, $t \in (0,a]$ and $x,y \in \mathbb{R}^d$ with $|x-y| \leq \sqrt{t}$:

$$\begin{aligned} \Gamma_t(x,y) &\geq p(t,x-y) - C(Xt^{\frac{1}{2}} \vee Yt^{\frac{1-\alpha}{2}}) \exp\Big(Ct[X^2 + Y^{\frac{2}{1-\alpha}}]\Big)\, p(ct,x-y) \\ &\geq (2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{2}} - C((X^2a)^{\frac{1}{2}} \vee (Y^{\frac{2}{1-\alpha}}a)^{\frac{1-\alpha}{2}}) \exp\Big(Ca[X^2 + Y^{\frac{2}{1-\alpha}}]\Big)\, c^{-\frac{d}{2}} (2\pi t)^{-\frac{d}{2}}. \end{aligned}$$

Therefore, it holds that $\Gamma_t(x,y) \geq \frac{1}{2}(2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{2}}$ if

$$C((X^2a)^{\frac{1}{2}} \vee (Y^{\frac{2}{1-\alpha}}a)^{\frac{1-\alpha}{2}}) \exp\Big(Ca[X^2 + Y^{\frac{2}{1-\alpha}}]\Big)\, c^{-\frac{d}{2}} \leq \frac{e^{-\frac{1}{2}}}{2}.$$

Hence there exists a $K \in (0,1)$ (which only depends on c, C and α) such that the choice $a = K[X^2 + Y^{\frac{2}{1-\alpha}}]^{-1}$ works. For this a we have (as $K \in (0,1)$ we use that $K \le K^{1-\alpha}$)

$$\begin{aligned} &C((X^2a)^{\frac{1}{2}} \vee (Y^{\frac{2}{1-\alpha}}a)^{\frac{1-\alpha}{2}}) \exp\left(Ca[X^2 + Y^{\frac{2}{1-\alpha}}]\right) c^{-\frac{d}{2}} \\ &\le CK^{\frac{1-\alpha}{2}} \exp\left(CK\right) c^{-\frac{d}{2}}. \end{aligned}$$

So by Lemma 4.1 there exist a $\kappa \in (0,1)$ and a $M > 1$ such that for all $t \in [0,\infty)$ and $x, y \in \mathbb{R}^d$,

$$\Gamma_t(x,y) \ge M^{-1-\frac{t}{a}} p(\kappa t, x-y) = \frac{1}{M} \exp\left(-t\frac{\log M}{K}[X^2 + Y^{\frac{2}{1-\alpha}}]\right) p(\kappa t, x-y).$$

This proves that (3) holds for a large enough C. □

5 Proof of Corollary 1.2

As before, we consider $b \in C([0,T], B^{-\alpha}_{\infty,1})$ for some $\alpha \in (0,\frac{1}{2})$ and we let $X = (X_t)_{t\in[0,T]}$ be the solution to the martingale problem for $((\mathscr{L}_t)_{t\in(0,T]}, \delta_x)$. We prove Corollary 1.2, which means that we estimate the probability that X escapes a box of size K before time T. The estimate is a consequence of our heat-kernel estimates (Theorem 1.1), Markov's inequality and the Garsia-Rademich-Rumsey inequality. By the latter (see Theorem 5.1 or [21, Theorem 2.1.3]) we have for $\kappa > 0$

$$\kappa|X_t - X_s| \le 4\int_0^{t-s} u^{-\frac{1}{2}} \sqrt{\log\left(1 + \frac{4(F_{T,\kappa} - T^2)}{u^2}\right)}\, \mathrm{d}u, \tag{31}$$

where

$$F_{T,\kappa} = \int_0^T \int_0^T \exp\left(\kappa\left(\frac{|X_{r_2} - X_{r_1}|}{|r_2 - r_1|^{\frac{1}{2}}}\right)^2\right) \mathrm{d}r_1\, \mathrm{d}r_2. \tag{32}$$

Indeed, with $\Psi(t) = e^{t^2} - 1$ (so that $\Psi^{-1}(s) = \sqrt{\log(1+s)}$), $p(t) = \sqrt{t}$ (so that $p(\mathrm{d}u) = \frac{1}{2}u^{-\frac{1}{2}}$) and $\phi = \kappa X$,

$$B := \int_0^T \int_0^T \Psi\left(\frac{|\phi(t) - \phi(s)|}{p(|t-s|)}\right) \mathrm{d}s\, \mathrm{d}t = F_{T,\kappa} - T^2,$$

so that

$$\kappa|X_t - X_s| \le 8\int_0^{t-s} \Psi^{-1}\left(\frac{4}{u^2}B\right) p(\mathrm{d}u)$$

gives (31) by Theorem 5.1.

Theorem 5.1 (Garsia-Rademich-Rumsey inequality)**.** *[21, Theorem 2.1.3] Let p and Ψ be continuous and strictly continuous functions on $[0, \infty)$ such that*

$$p(0) = \Psi(0) = 0, \qquad \lim_{t\to\infty} \Psi(t) = \infty.$$

Let $T > 0$ and $\phi \in C([0, T], \mathbb{R}^d)$. Then for $0 \leq s < t \leq T$

$$|\phi(t) - \phi(s)| \leq 8 \int_0^{t-s} \Psi^{-1} \left(\frac{4}{u^2} \int_0^T \int_0^T \Psi \left(\frac{|\phi(t) - \phi(s)|}{p(|t - s|)} \right) \mathrm{d}s \, \mathrm{d}t \right) p(\mathrm{d}u).$$

In the proof of Corollary 5.3 we will bound the right-hand side of (31) in terms of a function ζ. In the next lemma we start by gathering some auxiliary facts about ζ.

Lemma 5.2. *Let $\zeta, \psi \colon (0, \infty) \to (0, \infty)$ be given by*

$$\zeta(r) := \int_0^r u^{-\frac{1}{2}} \left(\sqrt{\log(1 + u^{-2})} \vee 1 \right) \mathrm{d}u, \qquad \psi(r) := r^{\frac{1}{2}} \sqrt{(\log(\tfrac{1}{r}) \vee 1)}.$$

There exist $m, M > 0$ such that $m\zeta(r) \leq \psi(r) \leq M\zeta(r)$ for all $r > 0$. Moreover, $\psi(rs) \leq \sqrt{2}\psi(r)\psi(s)$ for all $r, s > 0$ and ψ is strictly increasing.

Proof. That ψ is strictly increasing on (e, ∞) will be clear, whereas on $[0, e)$ it follows by calculating its derivative. Since ψ and ζ are continuous and bounded away from 0 and ∞ on compact subintervals of $(0, \infty)$, the existence of such m and M follows once we show that $\lim_{r\to 0} \frac{\zeta(r)}{\psi(r)}$ and $\lim_{r\to\infty} \frac{\zeta(r)}{\psi(r)}$ exist and are in $(0, \infty)$. By applying L'Hospital's rule we obtain

$$\lim_{r\to 0} \frac{\zeta(r)}{\psi(r)} = \lim_{r\to 0} \frac{\int_0^r u^{-\frac{1}{2}} \sqrt{\log(1 + u^{-2})} \, \mathrm{d}u}{r^{\frac{1}{2}} \sqrt{\log(\frac{1}{r})}} \in (0, \infty).$$

Indeed

$$\begin{aligned}
&\lim_{r\to 0} \frac{\int_0^r u^{-\frac{1}{2}} \sqrt{\log(1 + u^{-2})} \, \mathrm{d}u}{r^{\frac{1}{2}} \sqrt{\log(\frac{1}{r})}} \\
&= \lim_{r\to 0} \frac{r^{-\frac{1}{2}} \sqrt{\log(1 + r^{-2})}}{\frac{1}{2} r^{-\frac{1}{2}} \sqrt{\log(\frac{1}{r})} - \frac{1}{2} r^{\frac{1}{2}} \log(\frac{1}{r})^{-\frac{1}{2}} r^{-1}} = \lim_{r\to 0} \frac{\sqrt{\log(1 + r^{-2})}}{\frac{1}{2} \sqrt{\log(\frac{1}{r})} - \frac{1}{2} \log(\frac{1}{r})^{-\frac{1}{2}}}, \\
&= \lim_{a\to\infty} 2 \frac{\sqrt{\log(1 + a^2)}}{\sqrt{\log a} - \log(a)^{-\frac{1}{2}}} = 2 \sqrt{\lim_{a\to\infty} \frac{\log(1 + a^2)}{\log a}} = 2 \sqrt{\lim_{a\to\infty} \frac{2a^2}{1 + a^2}} = 2\sqrt{2}.
\end{aligned}$$

And also for $r \to \infty$ we have

$$\begin{aligned}
\lim_{r\to\infty} \frac{\zeta(r)}{\psi(r)} &= \lim_{r\to\infty} \frac{\int_0^{\sqrt{e-1}} u^{-\frac{1}{2}} \sqrt{\log(1 + u^{-2})} \, \mathrm{d}u + \int_{\sqrt{e-1}}^r u^{-\frac{1}{2}} \, \mathrm{d}u}{r^{\frac{1}{2}}} \\
&= \lim_{r\to\infty} r^{-\frac{1}{2}} \int_{\sqrt{e-1}}^r u^{-\frac{1}{2}} \, \mathrm{d}u = \lim_{r\to\infty} r^{-\frac{1}{2}} 2[\sqrt{r} - \sqrt{\sqrt{e - 1}}] = 2 \in (0, \infty).
\end{aligned}$$

Furthermore

$$\psi(rs) = (rs)^{\frac{1}{2}} \left(\sqrt{(\log(\tfrac{1}{r}) + \log(\tfrac{1}{s})) \vee 1} \right)$$

and for all $x, y \in \mathbb{R}$ we have $(x+y) \vee 1 \leq x \vee 1 + y \vee 1 \leq 2(x \vee 1)(y \vee 1)$. Therefore,

$$\psi(rs) \leq \sqrt{2}(rs)^{\frac{1}{2}} \left(\sqrt{\log(\tfrac{1}{r}) \vee 1} \right) \left(\sqrt{\log(\tfrac{1}{s}) \vee 1} \right) = \sqrt{2}\psi(r)\psi(s).$$

□

Corollary 5.3. *Let ψ be as in Lemma 5.2 and let $C > 0$ be as in Theorem 1.1. Then there exists an $M > 0$ such that for all $T \geq 1$*

in previous versions we put another factor in front of the exp, like $T^2 \wedge \|b\|^{-\frac{4}{1-\alpha}}$

$$\mathbb{E}_x \left[\exp \left(\frac{1}{M} \left(\sup_{\substack{s,t \in [0,T] \\ s<t}} \frac{|X_t - X_s|}{\psi(t-s)} \right)^2 \right) \right]$$
$$\leq M \exp \left(CT \left[\|\Delta_{-1} b\|^2_{C_T L^\infty} + \|\Delta_{\geq 0} b\|^{\frac{2}{1-\alpha}}_{C_T B^{-\alpha}_{\infty,1}} \right] \right). \tag{33}$$

Proof. The proof is inspired by [11, Corollary A.5]. Unfortunately we cannot directly apply that result, because the constant they derive depends on the time interval $[0, T]$ (even though this is not explicitly stated).

Let us define $G_{T,\kappa} := 2\sqrt{F_{T,\kappa} \vee 4}$, where $F_{T,\kappa}$ is as in (32). Let ζ be as in Lemma 5.2. By (31) and using $4(F_{T,\kappa} - T^2) \leq G^2_{T,\kappa}$ we have by a substitution and by Lemma 5.2 (observe that $G_{T,\kappa} \geq 4 \geq e$) that for $T \geq 1$, $\kappa > 0$, $s, t \in [0, T]$ with $s < t$ and by writing $G = G_{T,\kappa}$

$$\begin{aligned}
\kappa|X_t - X_s| &\leq 4 \int_0^{t-s} u^{-\frac{1}{2}} \sqrt{\log \left(1 + \frac{G^2}{u^2} \right)} \, \mathrm{d}u \\
&\leq 4\sqrt{G} \int_0^{\frac{t-s}{G}} u^{-\frac{1}{2}} \sqrt{\log \left(1 + \tfrac{1}{u^2} \right)} \, \mathrm{d}u \lesssim \sqrt{G}\zeta(\tfrac{t-s}{G}) \\
&\lesssim \sqrt{G}\psi(\tfrac{t-s}{G}) \lesssim \sqrt{G}\psi(t-s)\psi(\tfrac{1}{G}) \lesssim \psi(t-s)\sqrt{\log G}.
\end{aligned}$$

Let $M > 0$ be such that $\kappa|X_t - X_s| \leq \sqrt{M}\psi(t-s)\sqrt{\log G_{T,\kappa}}$ for all $T \geq 1$, $\kappa > 0$ and $s, t \in [0, T]$ with $s < t$. Then

$$\mathbb{E}_x \left[\exp \left(\frac{\kappa^2}{M} \left(\sup_{\substack{s,t \in [0,T] \\ s<t}} \frac{|X_t - X_s|}{\psi(t-s)} \right)^2 \right) \right] \leq \mathbb{E}_x[G_{T,\kappa}].$$

As by Jensen's inequality $\mathbb{E}_x[G_{T,\kappa}] = 2\mathbb{E}_x[\sqrt{F_{T,\kappa} \vee 4}] \leq 2\sqrt{\mathbb{E}_x[F_{T,\kappa}] + 4}$ we will obtain a bound of $\mathbb{E}_x[G_{T,\kappa}]$, by estimating $\mathbb{E}_x[F_{T,\kappa}]$. Let $c \in (0,1)$ and $\kappa > 0$ be such that $\kappa < \frac{1}{2c}$. Then for all $r_2, r_1 > 0$ with $r_2 \neq r_1$

$$\int_{\mathbb{R}^d} p(c|r_2 - r_1|, y) \exp(\kappa \Big(\frac{|y|}{|r_2 - r_1|^{\frac{1}{2}}} \Big)^2) \, \mathrm{d}y = (\tfrac{1}{1-2c\kappa})^{\frac{d}{2}} < \infty. \tag{34}$$

$$\int_{\mathbb{R}^d} p(ct,y)\exp(\kappa\frac{|y|^2}{t})\,\mathrm{d}y = (2\pi ct)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\exp\left((\kappa-\frac{1}{2c})\frac{|y|^2}{t}\right)\,\mathrm{d}y$$
$$= (2\pi(\tfrac{1}{\frac{1}{c}-2\kappa})t)^{-\frac{d}{2}}(\tfrac{1}{c(\frac{1}{c}-2\kappa)})^{\frac{d}{2}}\int_{\mathbb{R}^d}\exp\left(-\frac{|y|^2}{2(\frac{1}{\frac{1}{c}-2\kappa})t}\right)\,\mathrm{d}y = (\tfrac{1}{1-2c\kappa})^{\frac{d}{2}}.$$

Hence, by Theorem 1.1

$$\begin{aligned}\mathbb{E}_x[F_{T,\kappa}] &= \int_0^T\int_0^T \mathbb{E}_x\left[\exp(\kappa\left(\frac{|X_{r_2}-X_{r_1}|}{|r_2-r_1|^{\frac{1}{2}}}\right)^2)\right]\,\mathrm{d}r_1\,\mathrm{d}r_2\\ &= \int_0^T\int_0^T \mathbb{E}_x\left[\int_{\mathbb{R}^d}\Gamma_{|r_2-r_1|}(y,X_{r_1})\exp(\kappa\left(\frac{|y-X_{r_1}|}{|r_2-r_1|^{\frac{1}{2}}}\right)^2)\,\mathrm{d}y\right]\,\mathrm{d}r_1\,\mathrm{d}r_2\\ &\le C(\tfrac{1}{1-2c\kappa})^{\frac{d}{2}}\int_0^T\int_0^T\exp\left(C|r_2-r_1|\left[\|\Delta_{-1}b\|^2_{C_tL^\infty}+\|\Delta_{\ge 0}b\|^{\frac{2}{1-\alpha}}_{C_tB^{-\alpha}_{\infty,1}}\right]\right)\,\mathrm{d}r_1\,\mathrm{d}r_2.\end{aligned}$$

The proof is completed by observing that for $A\ge 1$

$$\int_0^T\int_0^T \exp(A|r_2-r_1|)\,\mathrm{d}r_1\,\mathrm{d}r_2 = 2\int_0^T\int_0^t e^{A(t-s)}\,\mathrm{d}s\,\mathrm{d}t \lesssim \frac{2}{A}\int_0^T e^{At}\,\mathrm{d}t \lesssim \frac{e^{AT}}{A^2}\le e^{AT}.$$

□

Proof of Corollary 1.2. As $T\ge 1\ge e^{-1}$ we have $\psi(T)=\sqrt{T}$. Therefore, by Markov's inequality for all $M,K>0$ and the fact that ψ is strictly increasing:

$$\begin{aligned}\mathbb{P}_x\Big(\sup_{t\in[0,T]}|X_t-x|\ge K\Big) &\le \mathbb{E}_x\Big[\exp\Big(\frac{1}{MT}\sup_{t\in[0,T]}|X_t-x|^2\Big)\Big]\exp\Big(-\frac{K^2}{MT}\Big)\\ &\le \mathbb{E}_x\Big[\exp\Big(\frac{1}{M}\Big(\sup_{\substack{s,t\in[0,T]\\ s<t}}\frac{|X_t-X_s|}{\psi(t-s)}\Big)^2\Big)\Big]\exp\Big(-\frac{K^2}{MT}\Big).\end{aligned}$$

So (4) follows from Corollary 5.3. □

A Appendix

Theorem A.1. *Suppose $\alpha<0$ and $\beta>0$ are such that $\alpha+\beta>0$. Let $p,p_1,p_2,q_1,q_2\in[1,\infty]$ be such that*

$$\tfrac{1}{p}=\min\{1,\tfrac{1}{p_1}+\tfrac{1}{p_2}\}. \tag{35}$$

For all $r\ge q_1$

$$\|u\cdot v\|_{B^\alpha_{p,r}}\lesssim \|u\|_{B^\alpha_{p_1,q_1}}\|v\|_{B^\beta_{p_2,q_2}}. \tag{36}$$

Proof. For the proof see also [18, Corollary 2.1.35]. By slightly adapting [2, Theorem 2.82] and by using the Hölder inequality and [2, Theorem 2.79] (for (38)), we obtain implies the following two estimates.

$$\|u \otimes v\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|u\|_{B^{\alpha}_{p_1,q_1}} \|v\|_{B^{\beta}_{p_2,q_2}}, \tag{37}$$

$$\|u \otimes v\|_{B^{\alpha}_{p,r}} \lesssim \|v\|_{L^{p_2}} \|u\|_{B^{\alpha}_{p_1,r}} \lesssim \|v\|_{B^{\beta}_{p_2,q_2}} \|u\|_{B^{\alpha}_{p_1,q_1}}. \tag{38}$$

As [2, Theorem 2.52] implies $\|u \odot v\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|u\|_{B^{\alpha}_{p_1,q_1}} \|v\|_{B^{\beta}_{p_2,q_2}}$, combining the above inequalities proves (36). □

Lemma A.2. *Let $\mathfrak{X}$ be a Banach space and $f : [0,\infty) \times [0,\infty) \to \mathfrak{X}$ be continuously differentiable and be such that $s \mapsto f(t-s,s)$ is Bochner integrable on $[0,t]$. Define $F(t) := \int_0^t f(t-s,s)\,\mathrm{d}s$. Then F is differentiable and*

$$\partial_t F(t) = f(0,t) + \int_0^t \mathrm{D}_1 f(t-s,s)\,\mathrm{d}s.$$

Proof. $F(t) = G(t,t)$, where $G(t,u) = \int_0^t f(u-s,s)\,\mathrm{d}s$. By the Lebesgue dominated convergence theorem for Bochner integrals, $\partial_u G(t,u) = \int_0^t \mathrm{D}_1 f(u-s,s)\,\mathrm{d}s$. The rest follows by the Fundamental law of Calculus and the chain rule (one may also want to use that Bochner integrable functions are Pettis integrable and their integrals agree). □

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