From Hamiltonian particle systems to Kinetic equations

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Plan of the lectures

1. Particle systems and BBKGY hierarchy (the paradigm of the Kinetic Theory)

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5. Rigorous derivation of the Boltzmann equation for short times

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7. The Landau equations: grazing collision limit
8. The Landau equations and weak-coupling limit

Particle systems and BBKGY hierarchy

N identical particles, unitary mass, in all the space $\mathbb{R}^3$.
Configurations in the phase space

$$z_1 \dots z_N = Z_N = (X_N, V_N) = (x_1 \dots x_N, v_1 \dots v_N)$$

$z_i = (x_i, v_i)$ denotes position and velocity of the $i - th$ particle.

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Two-body interaction $\varphi : \mathbb{R}^3 \to \mathbb{R}$, φ is spherically symmetric .

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = -\sum_{\substack{j=1 \dots N \\ j \neq i}} \nabla \varphi(x_i - x_j) \end{cases}$$

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N is very large. Statistical description.

Particle systems and BBKGY hierarchy

Probability measure $W^N(Z_N)dZ_N$ on $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$
Symmetry:

$$W^N(z_1 \dots z_i \dots z_j \dots z_N) = W^N(z_1 \dots z_j \dots z_i \dots z_N)$$

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The time evolved measure is defined by

$$W^N(Z_N; t) = W^N(\Phi^{-t}(Z_N))$$

$\Phi^t(Z_N)$ the flow with initial datum Z_N.
An evolution equation for $W^N(Z_N; t)$.

Particle systems and BBKGY hierarchy

$$\partial_t W^N(t) = \mathcal{L}_N W^N(t)$$

Particle systems and BBKGY hierarchy

$$\partial_t W^N(t) = \mathcal{L}_N W^N(t)$$

$$\begin{aligned}\mathcal{L}_N = & -\sum_{i=1}^{N}[v_i \cdot \nabla_{x_i} + F_i \cdot \nabla_{v_i}] \\ & -\sum_{i=1}^{N}(v_i \cdot \nabla_{x_i}) - \sum_{\substack{i,j=1 \\ i<j}}^{N} F(x_i - x_j) \cdot (\nabla_{v_i} - \nabla_{v_j})\end{aligned}$$

$$F_i = -\sum_{j:j\neq i} \nabla\varphi(x_i - x_j)$$

Particle systems and BBKGY hierarchy

We are interested in the limit $N \to \infty$. N particle description vs one-particle description.
Define the j-particle marginals

$$f_j^N(Z_j; t) = \int dz_{j+1} \dots dz_N W^N(Z_j, z_{j+1} \dots z_N; t) \qquad j = 1 \dots N.$$

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Evolution equation for $f_j(t)$.

$$\int dz_{j+1}\dots dz_N(\partial_t + \sum_{i=1}^{N} v_i\cdot\nabla_{x_i})W^N(Z_j, z_{j+1}\dots z_N) = (\partial_t + \sum_{i=1}^{j} v_i\cdot\nabla_{x_i})f_j^N$$

free term

Particle systems and BBKGY hierarchy

$$\int dz_{j+1} \dots dz_N \sum_{i=1}^{N} F_i \cdot \nabla_{v_i} W^N(Z_j, z_{j+1} \dots z_N) =$$

$$-\sum_{i=1}^{N} \sum_{k=1, k \neq i}^{N} \int dz_{j+1} \dots dz_N \nabla_{x_i} \varphi(x_i - x_k) \cdot \nabla_{v_i} W^N(Z_j, z_{j+1} \dots z_N),$$

Particle systems and BBKGY hierarchy

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Particle systems and BBKGY hierarchy

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$\sum_{i=j+1}^{N}\sum_{k=1,k\neq i}^{N}\dots = 0$. $\sum_{i=1}^{j}\sum_{k=1,k\neq i}^{j}\dots +$ the free term yields the j-particle Liouville operator

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$$(N - j) \sum_{i=1}^{j} \int dz_{j+1} \nabla_{x_i} \varphi(x_i - x_{j+1}) \cdot \nabla_{v_i} f_{j+1}^N(Z_j, z_{j+1}).$$

Particle systems and BBKGY hierarchy

Conclusion:

$$\partial_t f_j^N(t) = \mathcal{L}_j f_j^N(t) + (N-j) C_{j+1} f_{j+1}^N, \qquad j = 1 \dots N$$

Particle systems and BBKGY hierarchy

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$$C_{j+1} f_{j+1}^N(Z_j) = \sum_{i=1}^{j} \int dz_{j+1} \nabla_{x_i} \varphi(x_i - x_{j+1}) \cdot \nabla_{v_i} f_{j+1}^N(Z_j, z_{j+1})$$

$j < N$ and $C_{N+1} = 0$. For $j = N$ Liouville eq.n $f_N^N = W^N$.
BBKGY hierarchy.

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BBKGY hierarchy.
Interpretation.

Particle systems and BBKGY hierarchy

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BBKGY hierarchy.
Interpretation. Apparently not useful: we have in any case to solve Liouville.

Particle systems and BBKGY hierarchy

However if

$$f_2^N(x_1, v_1, x_2, v_2) = f_1^N(x_1, v_1) f_1^N(x_2, v_2)$$

we get a single nonlinear eq.n. Not true.

Particle systems and BBKGY hierarchy

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we get a single nonlinear eq.n. Not true. But, if for some reason

$$f_2^N(x_1, v_1, x_2, v_2) \approx f_1^N(x_1, v_1) f_1^N(x_2, v_2),$$

in some limiting situations...

Boltzmann equation

Boltzmann (1872) looked for an equation for

$$\mu(dz,t) = \frac{1}{N}\sum_{i=1}^{N}\delta(z - z_i(t))dz.$$

the empirical distribution

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the empirical distribution or for $f(x, v; t)$, the one-particle distribution.

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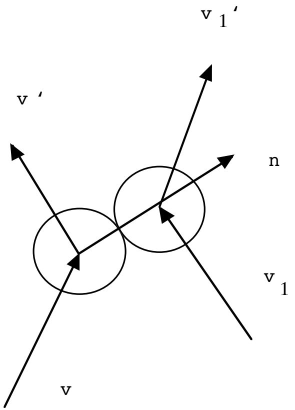
$$\mu(dz,t) = \frac{1}{N}\sum_{i=1}^{N} \delta(z - z_i(t))dz.$$

the empirical distribution or for $f(x,v;t)$, the one-particle distribution. The dynamics is that of hard spheres.

$$\begin{aligned} v' &= v - n[n\cdot(v-v_1)] \\ v_1' &= v_1 + n[n\cdot(v-v_1)]. \end{aligned}$$

Boltzmann equation



Boltzmann equation

The Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f)$$

Q is the collision operator

$$Q(f,f)(x,v) = \int_{\mathbb{R}^3} dv_1 \int_{S^2_+} dn \quad (v - v_1) \cdot n \quad [f(x,v')f(x,v_1') - f(x,v)f(x,v_1)]$$

n (the impact parameter) is a unitary vector and
$S^2_+ = \{n | n \cdot (v - v_1) \geq 0\}$

Boltzmann equation

Formal conservation in time of following five quantities

$$\int dx \int dv f(x, v; t) v^{\alpha}$$

with $\alpha = 0, 1, 2$, (conservation of the probability [mass], momentum and energy).

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Formal conservation in time of following five quantities

$$\int dx \int dv f(x, v; t) v^\alpha$$

with $\alpha = 0, 1, 2$, (conservation of the probability [mass], momentum and energy).
The entropy defined by

$$H(t) = \int dx \int dv\, f \log f(x, v, t) \tag{1}$$

is decreasing along the solutions. (The famous H-theorem)
Equilibrium.

Boltzmann equation

$\varphi = \varphi(v)$ a test function, consider the scalar product in $L^2(v)$:

$$(\varphi, Q(f,f)) = \int dv \int dv_1 \int_{S^2_+} dn \quad B(v - v_1; n) \quad \varphi[f' f_1' - f f_1].$$

Here we use the standard notation
$f = f(v), f' = f(v'), f_1 = f(v_1), f_1' = f(v_1')$.

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Here we use the standard notation
$f = f(v), f' = f(v'), f_1 = f(v_1), f'_1 = f(v'_1)$. Using the symmetry
$v \longrightarrow v_1$ and the fact that the Jacobian of

$$v, v_1 \longrightarrow v', v'_1$$

is unitary,

$$(\varphi, Q(f,f)) = \frac{1}{2} \int dv \int dv_1 \int_{S^2_+} dn \quad B(v - v_1; n) \quad f f_1 [\varphi' + \varphi'_1 - \varphi - \varphi_1]$$

The conservation of mass, momentum and energy follows

$$(\varphi, Q(f,f)) = 0$$

for $\varphi = 1, v, v^2$.

Boltzmann equation

H-theorem is consequence of the identity

$$\dot{H}(t) = (\log f, Q(f,f))$$

$\varphi = \log f,$

Boltzmann equation

H-theorem is consequence of the identity

$$\dot{H}(t) = (\log f, Q(f, f))$$

$\varphi = \log f$, we arrive to

$$(\log f, Q(f,f)) = \frac{1}{4}\int dx \int dv \int dv_1 \int_{S^2_+} dn \quad B(v - v_1; n)$$
$$[ff_1 - f'f_1'] \log \frac{f'f_1'}{ff_1}$$

Entropy production.
Boundary conditions.
Paradoxes.

The hard-sphere dynamics

N identical hard spheres of diameter ε of unitary mass.

$$Z_N = (z_1, \dots, z_N) = (X_N, V_N) = (x_1, \dots, x_N; v_1, \dots, v_N)$$

$z_i = (x_i, v_i) \in \mathbb{R}^6$.
The phase space

$$\Gamma_N^\varepsilon \subset (\mathbb{R}^6)^N =$$

$$\{Z_N \,|\, |x_i - x_j| \geq \varepsilon \quad \text{for } i \neq j\}$$

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Dynamics is the free flow
$Z_N(t) = (x_1 + v_1 t, \dots, x_N + v_N t; v_1, \dots, v_N)$ up to the first impact time when $|x_i - x_j| = \varepsilon$. Then an instantaneous collision takes place, according to the collision law and the flow goes on up to the next collision instant.

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The hard-sphere dynamics

Symmetric probability measure with density W_0^N on Γ_N^ε

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Symmetric probability measure with density W_0^N on Γ_N^ε

$$W^N(Z_N;t) = W_0^N(\Phi^{-t} Z_N).$$

The j-particle distributions f_j^N of a measure W^N (or marginals) are

$$f_j^N(z_1, \dots z_j; t) = \int dz_{j+1} \dots dz_N \quad W^N(z_1, \dots z_N; t)$$

Looking for an evolution equation for $f_j(t)$.

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Looking for an evolution equation for $f_j(t)$. H-S hierarchy.

H-S hierarchy

For $Z_N \in \Gamma_N^\varepsilon$

$$\frac{\partial}{\partial t} W^N(Z_N, t) = -\sum_{j=1}^{N} v_j \cdot \nabla_{x_j} W^N(Z_N, t).$$

H-S hierarchy

For $Z_N \in \Gamma_N^\varepsilon$

$$\frac{\partial}{\partial t} W^N(Z_N, t) = -\sum_{j=1}^{N} v_j \cdot \nabla_{x_j} W^N(Z_N, t).$$

Integrating on the last $N-1$ variables:

$$\frac{\partial}{\partial t} f_1(x, v; t) = -\sum_{j=1}^{N} \int_{\Gamma(x_1)} v_j \cdot \nabla_{x_j} W^N(x_1, v_1 \dots x_N, v_N) dx_2, dv_2 \dots dx_N, dv_N,$$

where

$$\Gamma(x_1) = \{x_2, v_2 \dots x_N, v_N \| x_1 - x_s | > \varepsilon,\ s = 2 \dots N\}.$$

H-S hierarchy

We deal first with the term $j = 1$

$$v_1 \cdot \nabla_{x_1} f_1(x_1, v_1) = v_1 \cdot \nabla_{x_1} \int_{\Gamma(x_1)} W^N(x_1, v_1 \dots x_N, v_N) dx_2, dv_2 \dots dx_N, dv_N$$
$$= \int_{\Gamma(x_1)} v_1 \cdot \nabla_{x_1} W^N dx_2, dv_2 \dots dx_N, dv_N +$$
$$\sum_{s=2}^{N} \int_{\partial\Gamma_s(x_1)} v_1 \cdot n_s W^N dx_2, dv_2 \dots d\sigma_s, dv_s \dots dx_N, dv_N$$

$$\partial\Gamma_s(x_1) = \{x_2, v_2 \dots x_N, v_N \,|\, |x_1 - x_r| > \varepsilon,\ r = 2 \dots N,\ r \neq s,\ |x_1 - x_s| = \varepsilon\}.$$

H-S hierarchy

By the div lemma:

$$\sum_{j=2}^{N} \int_{\Gamma(x_1)} v_j \cdot \nabla_{x_j} W^N(x_1, v_1 \dots x_N, v_N) dx_2, dv_2 \dots dx_N, dv_N$$

$$\sum_{j=2}^{N} \int_{\partial\Gamma_j(x_1)} v_j \cdot n_j W^N dx_2, dv_2 \dots d\sigma_j dv_j \dots dx_N, dv_N =$$

$$(N-1) \int dv_2 \int d\sigma_2 v_2 \cdot n_2 f_2(x_1, v_1, x_2, v_2) =$$

$$(N-1)\varepsilon^2 \int dv_2 \int dn v_2 \cdot n f_2(x_1, v_1, x_1 + \varepsilon n, v_2).$$

H-S hierarchy

Using the symmetry of W^N,

$$(\frac{\partial}{\partial t} + v_1 \cdot \nabla_{x_1}) f_1(x_1, v_1; t) = (N-1)\varepsilon^2 \int dv_2 (v_2 - v_1) \cdot n$$
$$\int dn\, f_2(x_1, v_1, x_1 + n\varepsilon, v_2)$$

H-S hierarchy

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again not very useful.

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again not very useful. The size of the r.h.s. $N\varepsilon^3 \approx 10^{-4} cm^3$, $N\varepsilon^2 = O(1)$

H-S hierarchy

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again not very useful. The size of the r.h.s. $N\varepsilon^3 \approx 10^{-4} cm^3$, $N\varepsilon^2 = O(1)$ the probability that two tagged particles collide is $O(\varepsilon^2)$. The probability that a given particle performs a collision is $O(N\varepsilon^2) = O(1)$.

$$f_2(x, v, x_2, v_2) = f_1(x, v) f_1(x_2, v_2)$$

statistical independence may be ok only **before** the collision.

H-S hierarchy

Applying factorization (propagation of chaos) only for the incoming velocities

$$f_2(x_1, v_1, x_1 + n\varepsilon, v_2) = f_1(x_1, v_1) f_1(x_1 + n\varepsilon, v_2)$$

if $(v_1 - v_2) \cdot n \geq 0$

H-S hierarchy

Applying factorization (propagation of chaos) only for the incoming velocities

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if $(v_1 - v_2) \cdot n \geq 0$ and

$$f_2(x_1, v_1, x_1 + n\varepsilon, v_2) = f_2(x_1, v_1', x_1 + n\varepsilon, v_2')$$

if $(v_1 - v_2) \cdot n < 0$ (here v_1', v_2' are incoming). Changing $n \to -n$, v_1', v_2' are outgoing.

H-S hierarchy

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if $(v_1 - v_2) \cdot n < 0$ (here v_1', v_2' are incoming). Changing $n \to -n$, v_1', v_2' are outgoing. We have the Boltzmann equation. However this can be true only in the limit

$$\varepsilon \to 0, \quad N \to \infty, \quad n\varepsilon^2 \to \text{const.}$$

called Boltzmann-Grad limit.

H-S hierarchy

In general we have

$$(\partial_t + \mathcal{L}_j^\varepsilon) f_j^N = (N-j)\varepsilon^2 C_{j+1}^\varepsilon f_{j+1}^N, \quad j = 1 \dots N$$

where $\mathcal{L}_j^\varepsilon$ is the generator of the dynamics of j hard-spheres of diameter ε.

$$C_{j+1}^\varepsilon f_{j+1}^N(x_1, v_1, \dots x_j, v_j) = -\sum_{k=1}^{j} \int dn \int dv_{j+1} n \cdot (v_k - v_{j+1})$$

$$f_{j+1}^N(x_1, v_1, \dots x_k, v_k, \dots, x_k + \varepsilon n, v_{j+1})$$

where n is the unit vector.
$f_j^N = 0$ if $j > N$ and hence, for $j = N$, we have nothing else than the Liouville equation.

H-S hierarchy

By the previous manipulations:

$$C^{\varepsilon}_{j+1} f^N_{j+1}(x_1, v_1, \dots x_j, v_j) = \sum_{k=1}^{j} \int dv_{j+1} \int_{S^+} dnn \cdot (v_k - v_{j+1})$$

$$f^N_{j+1}(x_1, v_1, \dots x_k, v'_k, \dots, x_k - \varepsilon n, v'_{j+1})$$

$$-f^N_{j+1}(x_1, v_1, \dots x_k, v_k, \dots, x_k + \varepsilon n, v_{j+1}) -$$

H-S hierarchy

$$C^{\varepsilon}_{j+1} = \sum_{k=1}^{j} C^{\varepsilon}_{k,j+1}$$

$$C^{\varepsilon}_{k,j+1} = C^{\varepsilon+}_{k,j+1} - C^{\varepsilon-}_{k,j+1}$$

and

$$C^{\varepsilon+}_{k,j+1} g_{j+1}(x_1, \dots, x_j; v_1, \dots, v_j) = \int dv_{j+1} \int_{S^2_+} d\omega \omega \cdot (v_k - v_{j+1})$$
$$[g_{j+1}(x_1, \dots, x_j, x_k - \varepsilon\omega; v_1, \dots, v'_k \dots, v'_{j+1}),$$

$$C^{\varepsilon-}_{k,j+1} g_{j+1}(x_1, \dots, x_j; v_1, \dots, v_j) = \int dv_{j+1} \int_{S^2_+} d\omega \omega \cdot (v_k - v_{j+1})$$
$$g_{j+1}(x_1, \dots, x_j, x_k + \varepsilon\omega; v_1, \dots, v_k, \dots, v_{j+1})].$$

H-S hierarchy

Therefore, by the Dyson (Duhamel) expansion:

$$f_j^{\varepsilon}(t) = \sum_{n=0}^{N-j} \sum_{\substack{\sigma_1,\dots,\sigma_n \\ \sigma_i = \pm 1}} \sum_{k_1,\dots,k_n}{}' (-1)^{|\underline{\sigma}_n|} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n$$

$$\alpha_n^{\varepsilon}(j) U^{\varepsilon}(t-t_1) C_{k_1,j+1}^{\varepsilon,\sigma_1} \dots U^{\varepsilon}(t_{n-1}-t_n) C_{k_n,j+n}^{\varepsilon,\sigma_n} U^{\varepsilon}(t_n) f_{0,n+j}^{\varepsilon}.$$

$$\alpha_n^{\varepsilon}(j) = \varepsilon^{2m}(N-j)(N-j-1)\dots(N-j-n+1).$$

H-S hierarchy

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$$\alpha_n^\varepsilon(j) = \varepsilon^{2m}(N-j)(N-j-1)\dots(N-j-n+1).$$

Here

$$\underline{\sigma}_n = (\sigma_1, \dots, \sigma_n), \quad \sigma_i = \pm \qquad |\underline{\sigma}_n| = \sum_j \sigma_j 1$$

and

$$\sum_{k_1,\dots,k_n}{}' = \sum_{k_1=1}^{j} \sum_{k_2=1}^{j+1} \cdots \sum_{k_n=1}^{j+n-1}$$