Low-rank techniques applied to moment equations for the stochastic Darcy problem with lognormal permeability

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Outline

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Darcy problem with log-normal permeability

We study the groundwater flow in a saturated heterogeneous medium where the permeability is described as a log-normal stochastic r.f. (model widely used in geophysical applications)

$$\begin{cases} -\mathrm{div}(e^{Y(\omega,x)}\nabla u(\omega,x)) = f(x), & \text{a.e. in } D \subset \mathbb{R}^d,\ d=1,2,3 \\ u(\omega,x) = g(x), & \text{a.e. on } \Gamma_D, \\ e^{Y(\omega,x)}\partial_n u(\omega,x) = h(x), & \text{a.e. on } \Gamma_N. \end{cases}$$

$$Y(\omega,x) : \text{Gaussian r.f.}, \quad \mathbb{E}[Y](x) = \mu(x), \quad \mathbb{C}ov[Y](x,y) = \rho(x,y),$$
$$\sigma := \left(\frac{1}{|D|}\int_D \rho(x,x)dx\right)^{\frac{1}{2}} < 1.$$

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Assumption: $\mathbb{C}ov\left[Y\right] \in \mathcal{C}^{0,t}(\overline{D \times D})$ for some $0 < t \leq 1$.

$$\Longrightarrow \quad Y \text{ a.s. continuous and } \|Y\|_{L^\infty(D)} \in L^p(\Omega),\ \forall p \geq 1$$

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Under the above assump. the prb. admits a unique solution $u \in L^p(\Omega; H^1(D)),\ \forall p \geq 1$. [Galvis – Sarkis, 2009, Gittelson, 2010, Charrier – Debussche, 2013]

Goal:

Compute statistical quantities for u, i.e. assess how the uncertainty on the permeability reflects on u.

- Expected value $\mathbb{E}[u](x) := \int_\Omega u(\omega, x) d\mathbb{P}(\omega)$
- Variance $\mathbb{V}ar[u](x) := \mathbb{E}\left[u^2\right](x) - \mathbb{E}[u]^2(x)$
- m-points correlation $\mathbb{E}[u^{\otimes m}](x_1, \ldots, x_m) := \mathbb{E}[u(\omega, x_1) \otimes \ldots \otimes u(\omega, x_m)]$

Method adopted:

Moment equations

Derive, theoretically analyze and numerically solve the deterministic equations solved by the statistical moments of the stochastic solution

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Perturbation approach and moment equations

(see e.g. Hydrology literature: [Tartakovsky – Neuman, 2008], [Riva – Guadagnini – De Simoni, 2006], Math. literature: [von Petersdorff – Schwab, 2006], [Todor PhD, 2005], [Harbrecht – Schneider – Schwab, 2008])

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Step 1.

Formally write the Taylor polynomial of $u(Y,x)$ w.r.t. Y, centered in $\mathbb{E}[Y]$.

$$u \simeq T^K u(Y,x) = \sum_{k=0}^{K} \frac{u^k(Y,x)}{k!},$$

$u^k = D^k[\mathbb{E}[Y]](Y,\ldots,Y)$
k-th Gateaux derivative of u

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- The k-th Gateaux derivative satisfies a recursive problem (for simplicity here $\mathbb{E}[Y]=0$)

$$\int_D \nabla u^k(x) \cdot \nabla v(x)\, dx = -\sum_{l=1}^{k} \binom{k}{l} \int_D Y^l(x) \nabla u^{k-l}(x) \cdot \nabla v(x)\, dx$$

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- The derivatives u^k are not directly computable (they are still ∞-dimensional random fields)

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$$\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right](x_1, \ldots, x_{l+1}) = \mathbb{E}\left[u^{k-l}(x_1) \otimes Y(x_2) \otimes \cdots \otimes Y(x_{l+1})\right]$$

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and evaluate it on the diagonal $(x, \dots, x) \in D^{\times(l+1)}$:

$$\mathbb{E}\left[\nabla u^{k-l}(x) Y^l(x)\right] = \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) \mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right](x, \dots, x)$$

Perturbation approach and moment equations

Step 3.

Write the recursion for the $(l+1)$-points correlations $\mathbb{E}\left[u^j \otimes Y^{\otimes l}\right], \quad j+l \leq k$.

We start from the problem solved by the k-th derivative:

$$\int_D \nabla u^j(x) \cdot \nabla v(x)\, dx = -\sum_{s=1}^{j} \left(\begin{array}{c} j \\ s \end{array} \right) \int_D Y^s(x) \nabla u^{j-s}(x) \cdot \nabla v(x)\, dx$$

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l.h.s. $= \int_D \nabla u^j(x_1) \cdot \nabla v(x_1)\, dx_1$

r.h.s. $= \int_D Y^s(x_1) \nabla u^{j-s}(x_1) \cdot \nabla v(x_1)\, dx_1$

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r.h.s. $= \int_D Y(x_2) \left(\int_D Y^s(x_1) \nabla u^{j-s}(x_1) \cdot \nabla v(x_1)\, dx_1 \right) v(x_2)\, dx_2$

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Step 3.

$$\int_{D^{\times(l+1)}} \nabla \otimes \mathrm{Id}^{\otimes l}\mathbb{E}\left[u^j \otimes Y^{\otimes l}\right] \cdot \nabla \otimes \mathrm{Id}^{\otimes l} v \; dx_1 \dots dx_{l+1}$$
$$= -\sum_{s=1}^{j} \binom{j}{s} \int_{D^{\times(l+1)}} \mathrm{Tr}_{|_{1:s+1}} \mathbb{E}\left[\nabla u^{j-s} \otimes Y^{\otimes(s+l)}\right] \cdot \nabla \otimes \mathrm{Id}^{\otimes l} v \; dx_1 \dots dx_{l+1}$$

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- This is a sequence of deterministic high dimensional problems.
- A similar recursion can be written for higher order moments. For instance, the k-th order correction to the second moment will involve the computation of all the correlations

$$\mathbb{E}[u^{j_1} \otimes u^{j_2} \otimes Y^{\otimes l}], \qquad j_1 + j_2 + l \leq k$$

The structure of the recursion for the first moment

Dim.	$k=0$	$k=1$	$k=2$	
d	u^0	$\mathbb{E}\left[u^1\right]$	$\mathbb{E}\left[u^2\right]$	$\cdots$
2d	$\mathbb{E}\left[u^0 \otimes Y\right]$	$\mathbb{E}\left[u^1 \otimes Y\right]$	$\iddots$	
3d	$\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right]$	$\iddots$		
	$\vdots$			

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The Algorithm

for $k = 0, \dots, K$
  Compute $\mathbb{E}\left[u^0 \otimes Y^{\otimes k}\right]$
  for $j = 1, \dots, k$
    Solve the boundary value problem for $\mathbb{E}\left[u^j \otimes Y^{\otimes k-j}\right]$
  end
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If $\mathbb{E}[Y] = 0$, $\mathbb{E}\left[Y^{\otimes(2k+1)}\right] = 0 \ \forall \ k$

A few relevant questions

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    - What is the accuracy of the Taylor approximation?
2. The k-th order correction to the mean (or higher order moments) can be obtained by solving the recursion for the correlations $\mathbb{E}[u^j \otimes Y^l]$, $j, l \leq k$.
    - Are these problems well posed?
    - What is the smoothness of the correlations functions $\mathbb{E}[u^j \otimes Y^l]$?

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    - Are these problems well posed?
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3. From the numerical point of view
    - How can we effectively approximate and solve the equations for the correlations $\mathbb{E}[u^j \otimes Y^l]$? (given that they are high dimensional objects)

Outline

Local convergence of the Taylor series

Let Y be a centered Gaussian random field ($\mathbb{E}[Y] = 0$). Consider the following map defined on the Banach space $L^\infty(D)$ with values in $H^1(D)$:

$$\begin{aligned} u : L^\infty(D) &\to H^1(D) \\ Y &\mapsto u(Y) \end{aligned}$$

and its Taylor polynomial $T^K u = \sum_{k=0}^{K} \frac{u^k}{k!}$, where $u^k = D^k[0](Y, \dots, Y)$.

Problem: Is the Taylor series $T^K u$ convergent in H^1-norm for $K \to +\infty$?

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$$\left\| T^K u \right\|_{H^1} \leq \sum_{k=0}^K \frac{\left\| u^k \right\|_{H^1}}{k!} \leq C \sum_{k=0}^K \left(\frac{\| Y \|_{L^\infty}}{\log 2} \right)^k$$

By a recursive argument we prove that $\left\| u^k \right\|_{H^1(D)} \leq C \left(\frac{\| Y \|_{L^\infty}}{\log 2} \right)^k k!$ with $C = C(C_P, \left\| u^0 \right\|_{H^1})$, C_P being the Poincaré constant.

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The Taylor series is convergent $\forall\ \sigma > 0$ in the disk $B := \{ Y \in L^\infty(D)\ :\ \| Y \|_{L^\infty} < \log 2 \}$

Global conv. of the Taylor series? A counter example

Let $Y(\omega, x) = \xi(\omega)x$, with $\xi \sim \mathcal{N}(0, \sigma^2)$ Gaussian random variable (one-dimensional probability space). Consider the following one-dimensional PDE

$$\begin{cases} -(\mathrm{e}^{\xi(\omega)x} u'(\omega, x))' = 0, & \text{a.e. in } [0,1] \\ u(\omega, 0) = 0, \ u(\omega, 1) = 1 \end{cases}$$

The exact solution is $u(\xi, x) = \frac{1-e^{-\xi x}}{1-e^{-\xi}}$.

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Let $Y(\omega, x) = \xi(\omega)x$, with $\xi \sim \mathcal{N}(0, \sigma^2)$ Gaussian random variable (one-dimensional probability space). Consider the following one-dimensional PDE

$$\begin{cases} -(e^{\xi(\omega)x} u'(\omega, x))' = 0, & \text{a.e. in } [0,1] \\ u(\omega, 0) = 0, \ u(\omega, 1) = 1 \end{cases}$$

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Observe that:

- On the real axis ($\xi \in \mathbb{R}$), $u(\xi, x)$ is analytic as a function of ξ.
- In the complex plane ($\xi \in \mathbb{C}$), $u(\xi, x)$ is not entire. Indeed, it admits countable many poles in $\xi = 2\pi i k$, $k \in \mathbb{Z} \setminus \{0\}$.

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The Taylor series centered in $\xi = 0$ converges only in the disk of radius $r < 2\pi$ and $\sum_{K \geq 0} \mathbb{E}\left[T^K u\right]$ is not convergent to $\mathbb{E}[u]$

A priori error upper bound

Given the counter example, in general we do not expect $\mathbb{E}\left[T^K u\right]$ to be convergent to $\mathbb{E}[u]$.

Nevertheless, for σ and K sufficiently small, $\mathbb{E}\left[T^K u\right]$ is a good approximation of $\mathbb{E}[u]$. The method we propose can be used even if the Taylor series is not globally convergent.

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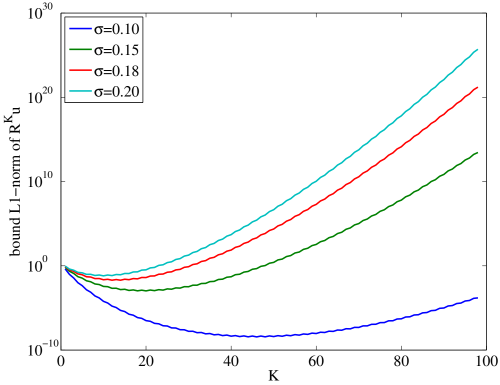
A priori error estimate $(0 < \sigma < 1)$ [Bonizzoni – Nobile, 2013]

$$\mathbb{E}\left\|u - T^K u\right\|_{H^1(D)} \leq C \frac{(K+1)!}{(\log 2)^{K+1}} \sum_{j=K+1}^{\infty} \frac{\sigma^j}{j!!} \leq C \left(\frac{\sigma}{\log 2}\right)^{K+1} K!!$$

Remark: $K!! = K(K-2)(K-4)\dots 1$

The error upper bound as a function of K



- Divergence of error upper bound $\forall\ \sigma > 0$

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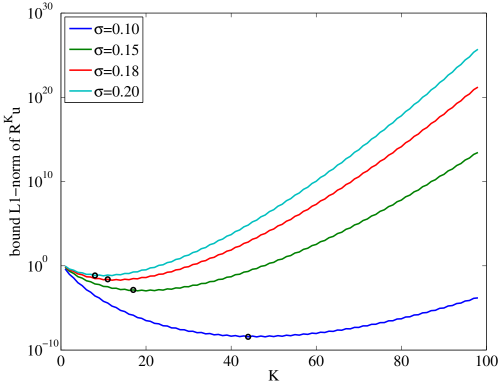

- Divergence of error upper bound $\forall\ \sigma > 0$
- Estimated "optimal" K, $K^{\sigma}_{opt} = \left\lfloor \left(\frac{\log 2}{\sigma}\right)^2 \right\rfloor - 4$. (bullets in the picture)

Error upper bound: sketch of the proof

Key ingredients:

1. We prove by a recursive argument that $\left\| u^k(tY,x) \right\|_{H^1(D)} \leq C e^{t\|Y\|_{L^\infty}} \left(\frac{\|Y\|_{L^\infty(D)}}{\log 2} \right)^k k!$, $0 \leq t \leq 1$
2. $\mathbb{E} \|Y\|^k_{L^\infty(D)} \leq C\ \sigma^k (k-1)!!$ (application of a result in [Adler – Taylor, 2007, Charrier – Debussche, 2013]).

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$$\mathbb{E}\left[\left\| u - T^K u \right\|_{H^1(D)} \right] \leq C(K+1)! \left(\frac{1}{\log 2} \right)^{K+1} \sum_{j=K+1}^{\infty} \frac{\mathbb{E}\left[\|Y\|^j_{L^\infty} \right]}{j!}$$

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$$\leq C(K+1)!\left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{\infty} \frac{\sigma^j}{j!!} \quad \text{[use (2)]}$$

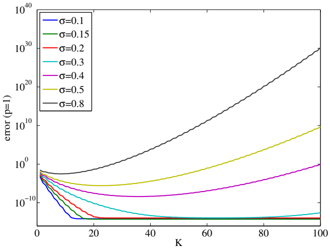
A numerical check: Single Gaussian random variable

$$-\mathrm{div}(e^{\cos(\pi x)\xi(\omega)}\nabla u(\omega,x)) = x \text{ a.e. in } D=[0,1]$$

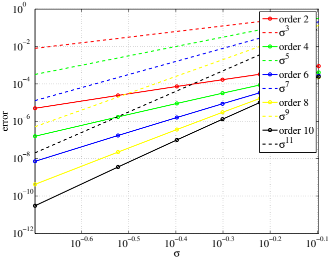
$$\xi(\omega)\sim\mathcal{N}(0,\sigma^2),\ 0<\sigma<1$$

The Taylor polynomial is computable!

Computed error vs K



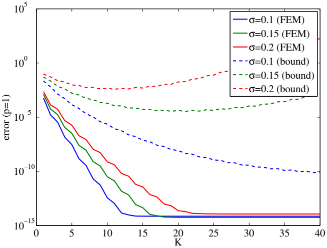
Computed error vs σ



- We numerically show the divergence of the Taylor series for any value of the standard deviation $\sigma > 0$
- The exponential behavior as function of σ is confirmed

How good is the a priori error estimate?

Comp. err. and err. estimate



- The a priori error bound is very pessimistic

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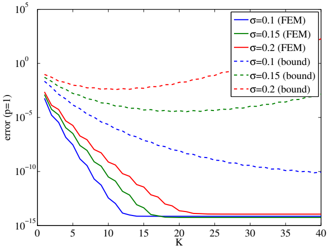
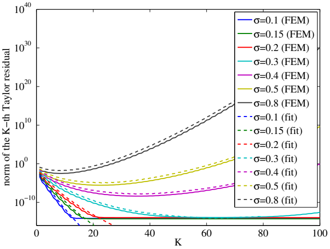


Comp. err. and fitted err. estimate



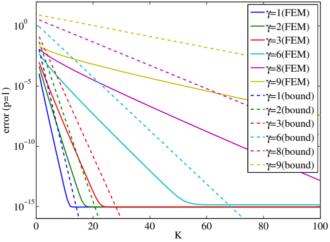
- The a priori error bound is very pessimistic
- It is possible to fit the parameter γ in the a priori error bound

$$\mathbb{E}\left\|u - T^K u\right\|_{H^1(D)} \leq C\left(\frac{\gamma\sigma}{\log 2}\right)^{K+1} K!!$$

Single bounded random variable

$$0 < \alpha_1 \leq a(\omega, x) = \mathbb{E}[a](x) + b(x)Y(\omega) \leq \alpha_2 < +\infty$$
$$Y(\omega) \subset [-\gamma, \gamma], \quad 0 < \gamma < +\infty$$

The Taylor series is convergent provided that the variability of a is small enough
[Babuška – Chatzipantelidis, 2002, Todor PhD, 2005]



Computed error and bound vs K

Outline

Moment equations: well-posedness and regularity results

Consider again the recursion for the correlations $\mathbb{E}[u^k \otimes Y^{\otimes l}]$:

Dim.	$k=0$	$k=1$	$k=2$	
d	u^0	0	$\mathbb{E}\left[u^2\right]$	$\cdots$
2d	0	$\mathbb{E}\left[u^1 \otimes Y\right]$	$\iddots$	
3d	$\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right]$	$\iddots$		
	$\vdots$			

Theorem: well-posedness [Bonizzoni PhD, 2013]

Let Y be a Gaussian random field with Gaussian covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, $0 < t \leq 1$. Then, all the problems in the recursion are well-posed.

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Theorem: regularity [Bonizzoni PhD, 2013]

Let Y be a Gaussian random field with Gaussian covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, $0 < t \leq 1$. Moreover, if the domain is convex and $\mathcal{C}^{1,t/2}$ and $u^0 \in \mathcal{C}^{1,t/2}(\bar{D})$, then $\mathbb{E}\left[u^k \otimes Y^{\otimes l}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times l}, \mathcal{C}^{1,t/2}(\bar{D})\right)$

Problem for $\mathbb{E}\left[u^1 \otimes Y\right]$ – Full TP discretization

Given $\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right] \in H^1(D) \otimes \left(L^2(D)\right)^{\otimes 2}$, find $\mathbb{E}\left[u^1 \otimes Y\right] \in H^1(D) \otimes L^2(D)$ s.t.

$$\int_D \int_D (\nabla \otimes \mathrm{Id})\, \mathbb{E}\left[u^1 \otimes Y\right](x_1, x_2) \cdot (\nabla \otimes \mathrm{Id})\, v(x_1, x_2)\ dx_1\ dx_2$$
$$= -\int_D \int_D \mathrm{Tr}_{|1:2} \mathbb{E}\left[\nabla u^0 \otimes Y^{\otimes 2}\right](x_1, x_2) \cdot (\nabla \otimes \mathrm{Id})\, v(x_1, x_2)\ dx_1\ dx_2$$

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Let us introduce:

$\{\phi_i\}_i$ linear FEM elements to discretize $H^1(D)$

$\{\psi_j\}_j$ piecewise constants to discretize $L^2(D)$

$A(n, m) = \int_D \nabla\phi_n(x)\nabla\phi_m(x)dx$
$M(i, j) = \int_D \psi_j(x)\psi_i(x)dx$
$\mathcal{B}^1(n, i_1, m) = \int_D \nabla\phi_n(x)\psi_{i_1}(x)\nabla\phi_m(x)dx$

$C_{1,1}(n, i)$ nodal repr. of $\mathbb{E}\left[u^1 \otimes Y\right]$
$C_{0,2}(n, i_1, i_2)$ nodal repr. of $\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right]$

$$A \otimes M\; C_{1,1} = -\mathcal{B}^1 \otimes M\; C_{0,2}$$

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Simplifying the mass matrix:

$$A \times_{1:1} \mathcal{C}_{1,1} = -\mathcal{B}^1 \times_{1:2} \mathcal{C}_{0,2}$$

where $\times_{1:s}$ denotes the saturation of the first s indices of both the right and left hand side tensors.

Problem for $\mathbb{E}\left[u^k \otimes Y^{\otimes l}\right]$ – Full TP discretization

Generalizing the previous equation:

Tensorial equation

$$A \times_{1:1} \mathcal{C}_{k,l} = -\sum_{s=1}^{k} \binom{k}{s} \mathcal{B}^s \times_{1:s+1} \mathcal{C}_{k-s,s+l}$$

Problem: Curse of the dimensionality. How to store all the tensors?

Dim.	$k=0$	$k=1$	$k=2$	
$\mathcal{O}(N_h)$	$\mathcal{C}_{0,0}$	0	$\mathcal{C}_{2,0}$	$\cdots$
$\mathcal{O}(N_h^2)$	0	$\mathcal{C}_{1,1}$	$\iddots$	
$\mathcal{O}(N_h^3)$	$\mathcal{C}_{0,2}$	$\iddots$		
	$\vdots$			

Outline

Tensor Train (TT) Format [Oseledets, 2011]

Generalization of the SVD decomposition of a matrix in more than 2 dimensions.
SVD of a matrix: let $X \in \mathbb{R}^{N_1 \times N_2}$ be a matrix

$$X(i_1, i_2) = \sum_{\alpha_1=1}^{r_1} G_1(i_1, \alpha_1) G_2(\alpha_1, i_2)$$

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Tensor Train (TT) Format: let $X \in \mathbb{R}^{N_1 \times \ldots \times N_n}$ be a tensor of order n

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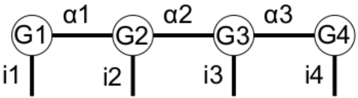
The $(n+1)$-tupla $(r_0, \ldots, r_n)$ is called $TT - rank$

Idea: Storage of order 3 tensors in a (linear) linked format



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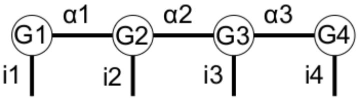


Pro:
- Storage complexity: $\mathcal{O}(nNr^2)$ vs $\mathcal{O}((N)^n)$, $r = \max r_i$, $N = \max N_i$.
- It allows fast computations.

Results obtained: Using the Matlab TT-toolbox 2.2 [Oseledets, 2012], we developed a code which solves the recursive problem for $\mathbb{E}[u]$ in TT-format.

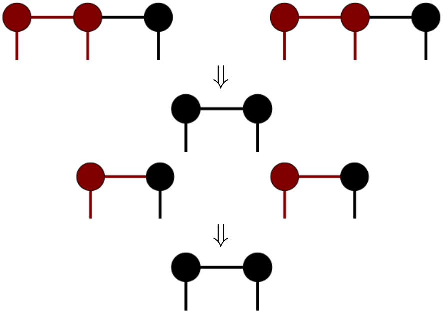
The TT–algorithm

What does it mean to solve a tensorial equation in TT-format?

$$A \times_{1:1} \mathcal{C}_{1,1} = -\mathcal{B}^1 \times_{1:2} \mathcal{C}_{0,2}$$
$$\Downarrow$$
$$\mathcal{C}_{1,1} = -A^{-1} \times_{1:1} \left(\mathcal{B}^1 \times_{1:2} \mathcal{C}_{0,2}\right)$$

STEP1
saturation
$\mathcal{B}^1 \times_{1:2} \mathcal{C}_{0,2}$

STEP2
saturation
with A^{-1}



$$A \times_{1:1} \mathcal{C}_{k,l} = -\sum_{s=1}^{k} \binom{k}{s} \mathcal{B}^s \times_{1:s+1} \mathcal{C}_{k-s,s+l} \tag{1}$$

Inputs needed:

- TT-format of the correlation $\mathcal{C}_{0,s}$, $\mathcal{C}_{0,s}^{TT}$, (nodal representation of $\mathbb{E}\left[u^0 \otimes Y^{\otimes s}\right]$)
- TT-format of the tensors $\mathcal{B}^s$, $\mathcal{B}^{TT,s}$
- Stiffness matrix A

Operations needed:

- Saturation $\times_{1:s}$ between two TT-tensors
- Lin. alg. operations and approximation (tt_round) of tt-tensors [TT-toolbox]

> **for** $k = 0, \dots, K$
> Compute $\mathcal{C}_{0,k}^{TT}$ with a tolerance tol_{TT}
> **for** $l = k-1, \dots, 0$
> Solve the tensorial equation (1)
> **end**
> The result for $l = 0$ is the k-th order correction $\mathcal{C}_{k,0}$
> **end**

TT-representation of $\mathbb{E}\left[Y^{\otimes k}\right]$ [Kumar – Kressner – Nobile – Tobler, 2013]

The starting point is the KL-expansion of the Gaussian random field Y:

$$Y(\omega, x) = \mathbb{E}\left[Y\right](x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\omega)\phi_i(x),\ x \in D,\ \omega \in \Omega$$

where ξ_i i.i.d $\sim \mathcal{N}(0,1)$ and $\sum_{i=1}^{\infty} \lambda_i = \int_D Var[Y(x)]dx$.

- k-th correlation:
  $\mathbb{E}\left[Y^{\otimes k}\right](x_1 \dots, x_k) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \mathbb{E}\left[\prod_{\eta=1}^{k} \sqrt{\lambda_{i_\eta}}\xi_{i_\eta}\right] \bigotimes_{\eta=1}^{k} \phi_{i_\eta}(x_\eta) =$
  $\sum_{\mathbf{i}\in\mathbb{N}^k} C_{i_1\dots i_k} \bigotimes_{\eta=1}^{k} \phi_{i_\eta}(x_\eta)$,
  where $C_{i_1\dots i_k} = \prod_{l=1}^{\infty} \lambda_l^{m_{\mathbf{i}}(l)/2}\mathbb{E}\left[\xi_l^{m_{\mathbf{i}}(l)}\right]$, $m_{\mathbf{i}}(l)$=multiplicity of index l in $\mathbf{i}$.
- $C_{i_1\dots i_k}$ is supersymmetric.
- An exact TT symmetric representation can be constructed:

$$C^{(1,\dots,k/2)} = U_{k/2} M U_{k/2}^T$$

  with $U_{k/2}$ basis of $Range(C^{(1,\dots,k/2)})$.
- Then the basis $C^{(1,\dots,k/2)}$ can be further truncated with a given tolerance tol_{TT}:

$$\left\| C - \widetilde{C} \right\|_F \leq tol_{TT}$$

Outline

Test 1 – Analysis of the Taylor approximation

Let $Y(\omega, x)$ be a centered Gaussian r. f. with Gaussian cov. function

$$Cov_Y(x_1, x_2) = \sigma^2 \, e^{-\frac{\|x_1 - x_2\|^2}{0.2^2}}, \quad (x_1, x_2) \in [0,1] \times [0,1]$$

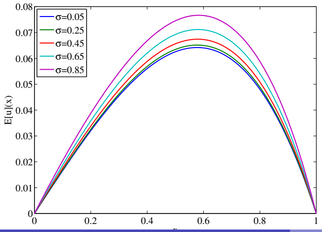
To compute the reference solution and the TT-solution we use:

- same spatial discretization $N_h = 100$ of the physical domain $D = [0,1]$
- same KL-expansion: $N = 11$ r.v. (99% of variance captured)
- exact TT-computations: $tol_{TT} = 10^{-16}$

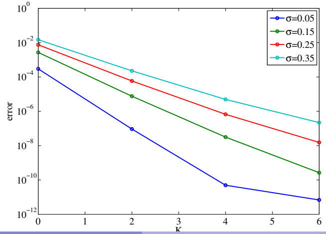
$\Downarrow$

We observe only the truncation error in the Taylor series

Reference solution (collocation)

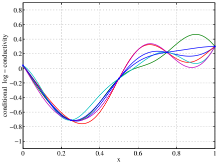


Computed error vs k



Test 1 – Analysis of the Taylor approx.

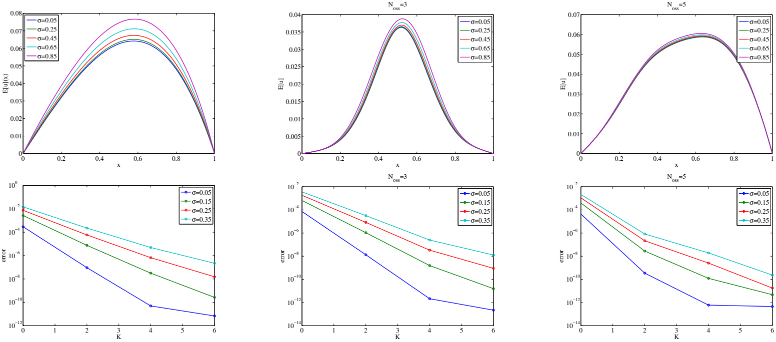
- N_{obs} = number of observations of the permeability field
- N = number of random variables considered



$N_{obs} = 0,\ N = 11$ | $N_{obs} = 3,\ N = 9$ | $N_{obs} = 5,\ N = 8$

mean coll.

error vs K

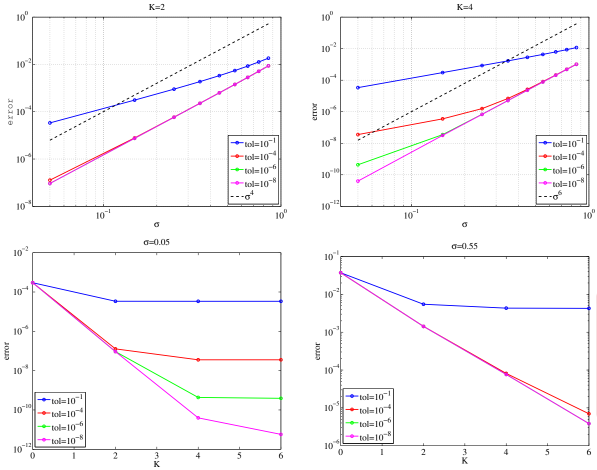


As N_{obs} increases, the variability of the field decreases: good for perturbation methods!

Test 2 – Analysis of the dependence on the TT-precision

Let $Y(\omega, x)$ be a centered Gaussian r. f. with Gaussian cov. function

- same spatial discretization $N_h = 100$ of the physical domain $D = [0, 1]$
- same KL-exp: $N = 26$ r.v. (100% of variance captured up to machine precision)
- different tolerances in the TT-computations

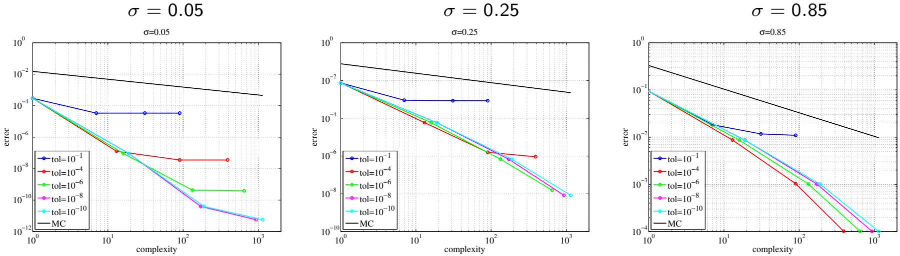


It is not always useful to consider small tol_{TT}.
There is an optimal tol_{opt} depending on σ and K

Test 2 – The complexity of the TT-algorithm

Complexity= number of linear systems to be solved

We numerically studied how the error depends on the complexity of the TT-algorithm



error vs complexity for different tol_{TT}

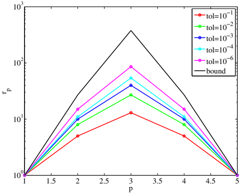
If the optimal tol_{opt} is chosen, the TT-algorithm is far superior to a standard Monte Carlo method (black line)

Test 3 – Storage requirements of the TT-algorithm

Let $Y(\omega, x)$ be a centered Gaussian r. f. with Gaussian cov. function

- spatial discretization $N_h = 200$ of the physical domain $D = [0, 1]$
- exact KL-exp: $N = 27$ r.v. (100% of variance captured up to machine precision)
- different tolerances in the TT-computations

TT-ranks of the TT-correlations $\mathcal{C}^{TT}_{Y^{\otimes k}}$ for different tol_{TT}

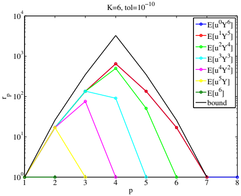


$$r_p \leq \binom{N+p-1}{p}$$

(black line)

[Kumar – Kressner – Nobile – Tobler]

TT-ranks of the correlations in the recursion for $tol_{TT} = 10^{-10}$



the upper bound (black line) is valid

The storage requirement is a limiting aspect of our algorithm.

Improvements could be obtained thanks to the implementation of sparse TT-tensors

Conclusions

- We have applied the perturbation technique to the Darcy problem with lognormal permeability.
- We have studied the approximation properties of the Taylor polynomial
- We have derived the moment equations, and proved their well-posedness and Hölder-type regularity results.
- We have developed an algorithm in TT-format able to solve the first statistical moment problem. Our TT-algorithm provide a valid solution both in the case where Y is parametrized by a small number of r.v. and if the entire random field is considered.
- If the optimal tol_{TT} is considered, our TT-algorithm is far superior to a standard Monte Carlo method
- The main limitation is the storage requirement.

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Thank you for the attention!

Well-posedness of the stochastic Darcy problem

$$\text{find } u \in L^p\left(\Omega; H^1(D)\right) \text{ s.t. } u|_{\Gamma_D} = g \text{ a.s., and}$$
$$\int_D a(\omega,x)\nabla_x u(\omega,x)\cdot\nabla_x v(x)\,dx = \int_D f(x)v(x)\,dx \quad \forall v \in H^1_{\Gamma_D}(D), \text{ a.s. in } \Omega.$$

A1 : *The permeability field* $a \in L^p\left(\Omega; C^0(\bar{D})\right)$ *for every* $p \in (0,\infty)$.

Then, the quantities

$$a_{min}(\omega) := \min_{x\in\bar{D}} a(\omega,x) \tag{2}$$
$$a_{max}(\omega) := \max_{x\in\bar{D}} a(\omega,x) \tag{3}$$

are well defined, and $a_{max} \in L^p(\Omega)$ for every $p \in (0,+\infty)$. Moreover, we assume

A2 : $a_{min}(\omega) > 0$ *a.s.*, $\dfrac{1}{a_{min}(\omega)} \in L^p(\Omega)$ *for every* $p \in (0,\infty)$.

Theorem

If the permeability field $a(\omega,x)$ *satisfies* **A1**, **A2**, *then the stochastic Darcy problem is well-posed for every* $p \in (0,\infty)$, *that is it admits a unique solution that depends continuously on the data.*

Upper bounds for the statistical moments of $\|Y\|_{L^\infty(D)}$

KL expansion: $Y(\omega,x) = \mathbb{E}[Y](x) + \sigma^2 \sum_{j=1}^{+\infty} \sqrt{\widetilde{\lambda}_j}\, \phi_j(x)\, \xi_j(\omega)$

1. ϕ_j is Hölder continuous with exponent $0 < \gamma \leq 1$ for every $j \geq 1$.
2. $R_\gamma := \sum_{j=1}^{+\infty} \widetilde{\lambda}_j \|\phi_j\|^2_{\mathcal{C}^{0,\gamma}(\bar{D})} < +\infty$.

Spectral technique [Charrier – Debussche, 2013]

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq C\, R_\gamma^{k/2} \sigma^k\, (k-1)!!, \quad \forall k > 0$$

The domain is a d-dimensional rectangle $D = [0, T]^d$. The centered Gaussian field $Y'(\omega,x)$ is stationary and regular ($\mathcal{C}^2$)

Euler characteristic technique [Adler – Taylor, 2007]

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq C\, \sigma^{k-2}\, k\, (k-1)!!, \quad \forall k$$

Problem solved by $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$

$$\int_D \dots \int_D \nabla \otimes \mathrm{Id}^{\otimes l} \mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \cdot \nabla \otimes \mathrm{Id}^{\otimes l} v \; dx_1 \dots dx_{l+1} =$$
$$-\sum_{s=1}^{k-l} \binom{k-l}{s} \int_D \dots \int_D \mathbb{E}\left[(\nabla u^{k-l-s} Y^s) \otimes Y^{\otimes l}\right] \cdot \nabla \otimes \mathrm{Id}^{\otimes l} v \; dx_1 \dots dx_{l+1}$$

Hölder spaces with mixed regularity

$\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})$, $0 < \gamma \leq 1$, is the space of all cont. funct. $v : \bar{D}^{\times k} \to \mathbb{R}$ s.t.

$$|v|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} := \sup_{\substack{\mathbf{x},\mathbf{x}+\mathbf{h}\in\bar{D}^{\times k} \\ \mathbf{h}>0}} \left| D_{\mathbf{h}}^{\gamma,mix} v(x_1,\ldots,x_k) \right| < +\infty,$$

where

$$D_{\mathbf{h}}^{\gamma,mix} v(x_1,\ldots,x_k) := D_{1,h_1}^{\gamma} \cdots D_{k,h_k}^{\gamma} v(x_1,\ldots,x_k),$$

with

$$D_{i,h_i}^{\gamma} v(x_1,\ldots,x_k) := \frac{v(x_1,\ldots,x_i+h_i,\ldots,x_k) - v(x_1,\ldots,x_k)}{|h_i|^{\gamma}}.$$

$\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})$ is a Banach space with the norm

$$\|v\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} := \|v\|_{\mathcal{C}^{0}(\bar{D}^{\times k})} + |v|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})}.$$

- $\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k}) \subset \mathcal{C}^{0,\gamma}(\bar{D}^{\times k})$
- $\mathcal{C}^{0,\gamma}(\bar{D}^{\times k}) \subset \mathcal{C}^{0,\gamma/k,mix}(\bar{D}^{\times k})$

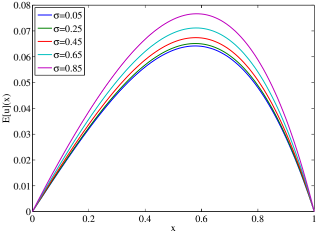
Gaussian cov. function – Truncated KL – error vs σ

Let $Y(\omega, x)$ be a centered Gaussian r. f. with Gaussian cov. function

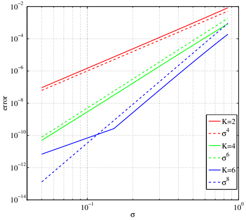
$$Cov_Y(x_1, x_2) = \sigma^2 \, e^{-\frac{\|x_1 - x_2\|^2}{0.2^2}}, \quad (x_1, x_2) \in [0,1] \times [0,1]$$

- $tol_{KL} = 10^{-4}$: $N_h = 100$, $N = 11$ r.v. (99% of variance captured)
- $tol_{TT} = 10^{-16}$

Reference solution (collocation)



Computed error vs σ



Order of $\left\|\mathbb{E}\left[u(Y,x)\right] - \mathbb{E}\left[T^K u(Y,x)\right]\right\|_{L^2(D)}$ as function of σ

	$K=0$	$K=1$	$K=2$	$K=3$	$K=4$	$K=5$	$K=6$
$\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2}$	2	2	4	4	6	6	8

Exponential cov. function – Complete KL

Let $Y(\omega, x)$ be a centered Guassian r. f. with exponential cov. function

$$Cov_Y(x_1, x_2) = \sigma^2 \, e^{-\frac{\|x_1 - x_2\|}{0.2}}, \quad (x_1, x_2) \in [0,1] \times [0,1]$$

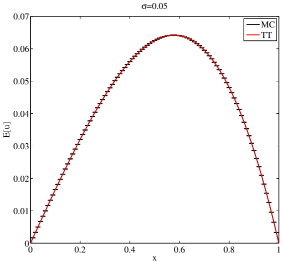
- $tol_{KL} = 10^{-4}$: $N_h = 100$, $N = 100$ r.v. (100% of variance captured)
- $tol_{TT} = 10^{-16}$

$\Downarrow$

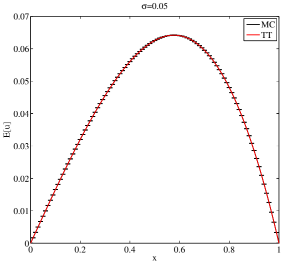
The collocation method is unusable.

We compare the TT-solution with the Monte Carlo solution (M=10000 samples)

$\sigma = 0.05$

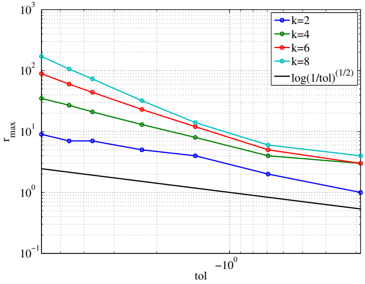


$\sigma = 0.65$

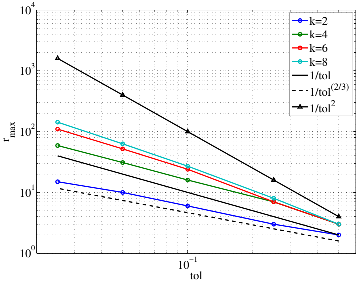


Dependence of the TT-ranks on the dimension

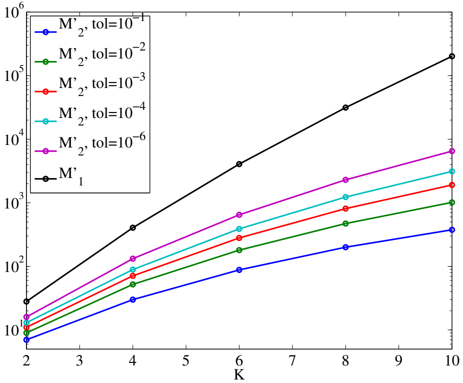
Gauss. Cov. funct.



Esp. Cov. funct.



Comparison with the comp. of a truncated Taylor series



Truncated Taylor expansion: $M_1' = \begin{pmatrix} N + K/2 \\ K/2 \end{pmatrix}$

TT-algorithm: $M_2' = \sum_{n=2:2:K} \sum_{p=0}^{n-1} r_p + 1$