Weierstraß-Institut für
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Leibniz Association

Advances in Adaptive SGFEM

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based on joint work with C.J.Gittelson, C.Merdon, C.Schwab, E.Zander

Elliptic BVP on Lipschitz $D \subset \mathbb{R}^d$, $u \in L^2_\pi(\Gamma) \otimes H^1_0(D)$,

$$-\nabla \cdot (a \nabla u) = f \quad \text{in } \Gamma \times D \quad \text{and} \quad u|_{\partial D} = 0.$$

Expansion of random field a

$$a(y, x) = a_0(x) + \sum_{m=1}^{\infty} y_m a_m(x)$$

with independent and uniformly distributed random variables

$$y = (y_m)_{m=1}^{\infty} \in \Gamma := [-1, 1]^{\infty}$$

and

$$\sum_{k=1}^{\infty} \left\| \frac{a_m}{a_0} \right\|_{L^\infty(D)} < 1, \quad a_0 > 0, \quad a_0, 1/a_0 \in L^\infty(D).$$

Stochastic basis

- determined by $\mathcal{F} := \{\mu \in \mathbb{N}_0^\infty : \#\operatorname{supp}\mu < \infty\}$
- for $\mu \in \mathcal{F}$, tensorised Legendre polynomials

$$P_\mu(y) := \prod_{m \in \operatorname{supp}\mu} P_{\mu_m}(y_m)$$

  form basis of $L^2_\pi(\Gamma)$

- three-term recursion of orthogonal polynomials

$$y_m P_\mu(y) = \beta_{\mu_m+1} P_{\mu+\epsilon_m}(y) + \beta_{\mu_m} P_{\mu-\epsilon_m}(y)$$

  with $(\epsilon_m)_n = \delta_{mn}$

Action of $A : v \mapsto -\nabla \cdot (a(y)\nabla v)$ on $u = (u_\mu)_{\mu \in \mathcal{F}}$ takes the form

$$(Au)_\nu = A_0 u_\nu + \sum_{m=1}^{\infty} A_m (\beta_{\mu_m+1} u_{\nu+\epsilon_m} + \beta_{\nu_m} u_{\nu-\epsilon_m})$$

for $\nu \in \mathcal{F}$ with

$$A_0 v := -\nabla \cdot (a_0 \nabla v) \quad \text{and} \quad A_m v := -\nabla \cdot (a_m \nabla v).$$

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for $\nu \in \mathcal{F}$ with

$$A_0 v := -\nabla \cdot (a_0 \nabla v) \quad \text{and} \quad A_m v := -\nabla \cdot (a_m \nabla v).$$

Expansion of solution

$$u(y, x) = \sum_{\mu\in\mathcal{F}} u_\mu(x) P_\mu(y)$$

where coefficients $u_\mu \in H_0^1(D)$ satisfy

$$Au = f.$$

$\mathbf{A}\mathbf{u} = \mathbf{f}$ is represented by

$$\begin{bmatrix} \ddots & & & \vdots & & & \iddots \\ & A_{\mu-\epsilon_m} & \cdots & B^{\mu}_{\mu-\epsilon_m,m,1} & \cdots & 0 & \\ & \vdots & & \vdots & & \vdots & \\ \cdots & B^{\mu-\epsilon_m}_{\mu,m,0} & \cdots & A_{\mu} & \cdots & B^{\mu+\epsilon_m}_{\mu,m,1} & \cdots \\ & \vdots & & \vdots & & \vdots & \\ & 0 & \cdots & B^{\mu}_{\mu+\epsilon_m,m,0} & \cdots & A_{\mu+\epsilon_m} & \\ \iddots & & & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ u_{\mu-\epsilon_m} \\ \vdots \\ u_{\mu} \\ \vdots \\ u_{\mu+\epsilon_m} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ f_{\mu-\epsilon_m} \\ \vdots \\ f_{\mu} \\ \vdots \\ f_{\mu+\epsilon_m} \\ \vdots \end{bmatrix}$$

with a basis $\{\varphi_j\}_{j=1}^{\infty}$ of $H_0^1(D)$ and

$$\left[A_\mu\right]_{ij} = \langle A_0 \varphi_i^{\mu}, \varphi_j^{\mu} \rangle \quad \text{and} \quad \left[B^{\mu_2}_{\mu_1,m,c}\right]_{ij} = \beta^m_{\mu_m+c} \langle A_m \varphi_i^{\mu_1}, \varphi_j^{\mu_2} \rangle$$

For a finite set $\Lambda \subset \mathcal{F}$ and the Galerkin projection $u_\Lambda = \sum_{\mu \in \Lambda} u_{\Lambda,\mu}(x) P_\mu(y) \in \mathcal{V}_\Lambda$, define the residual

$$r(u_\Lambda) := A(u - u_\Lambda) = f - A u_\Lambda.$$

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Then, for ν in boundary $\partial\Lambda \subset \mathcal{F} \setminus \Lambda$, i.e. $\nu = \mu \pm \epsilon_m$ for $\mu \in \Lambda$,

$$r_\nu(u_\Lambda) = \sum_{m=1}^{\infty} A_m(\beta_{\nu_m+1} u_{\Lambda,\nu+\epsilon_m} + \beta_{\nu_m} u_{\Lambda,\nu-\epsilon_m})$$

$$\|(r_\nu)_{\nu\in\partial\Lambda}(u_\Lambda)\| \le \zeta(u_\Lambda, \partial\Lambda) := \left(\sum_{\nu\in\partial\Lambda} \zeta_\nu(u_\Lambda)^2 \right)^{1/2}$$

with upper bound

$$\zeta_\nu(u_\Lambda) := \sum_{m=1}^{\infty} \left\| \frac{a_m}{a_0} \right\|_{L^\infty(D)} (\beta_{\nu_m+1} \|u_{\Lambda,\nu+\epsilon_m}\| + \beta_{\nu_m} \|u_{\Lambda,\nu-\epsilon_m}\|).$$

The Galerkin projection $u_\Lambda \in \prod_{\mu\in\Lambda} H_0^1(D)$

$$\langle A u_\Lambda, v_\Lambda \rangle = \langle f, v_\Lambda \rangle \qquad \forall\, v_\Lambda \in \prod_{\mu\in\Lambda} H_0^1(D)$$

yields the equivalence

$$\|u - u_\Lambda\|_A \approx \|(r_\nu)_{\nu\in\partial\Lambda}(u_\Lambda)\|_{A^*}$$

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$$\|u - u_\Lambda\|_A \approx \|(r_\nu)_{\nu\in\partial\Lambda}(u_\Lambda)\|_{A^*} \leq \zeta(u_\Lambda, \partial\Lambda).$$

Indicators ζ_ν can be used to enlarge Λ by $\Theta \subset \partial\Lambda$ s.t.

$$\sum_{\nu\in\Theta} \zeta_\nu(u_\Lambda) \geq \vartheta \sum_{\nu\in\partial\Lambda} \zeta_\nu(u_\Lambda) \qquad 0 < \vartheta \leq 1.$$

For

- some simplicial mesh $\mathcal{T}$ of D
- elements $T \in \mathcal{T}$ and edges $E \in \mathcal{E}$
- edge jump $[v]_E$ and normals n_E for $E \in \mathcal{E}$
- polynomial degree p

let $\mathcal{V}_p(\mathcal{T}) \subset H_0^1(D)$ denote the conforming space of piecewise polynomials of degree p on $\mathcal{T}$.

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let $\mathcal{V}_p(\mathcal{T}) \subset H_0^1(D)$ denote the conforming space of piecewise polynomials of degree p on $\mathcal{T}$.

A fully discrete approximation of u with

- finite element space $V_p(\mathcal{T})$ and
- index set $\Lambda \subset \mathcal{F}$

is given by

$$u_N(y,x) = \sum_{\mu\in\Lambda} u_{N,\mu}(x) P_\mu(y), \quad u_N = (u_{N,\mu})_{\mu\in\Lambda} \in \prod_{\mu\in\Lambda} V_p(\mathcal{T}).$$

Residual

$$\langle r_\mu(u_N), v\rangle := \int_D f\delta_{\mu 0} v - \sigma_{N,\mu}\cdot\nabla v \, dx$$ with numerical flux

$$\sigma_{N,\mu} := a_0 \nabla u_{N,\mu} + \sum_{m=1}^{\infty} a_m \nabla(\beta_{\mu_m+1} u_{N,\mu+\epsilon_m} + \beta_{\mu_m} u_{N,\mu-\epsilon_m}).$$

Local error estimator for $T \in \mathcal{T}$

$$\eta_T(u_N,\Lambda)^2 := \sum_{\mu\in\Lambda}\left(h_T^2 \|f\delta_{0\mu} + \nabla\cdot\sigma_{N,\mu}\|_T^2 + h_T \|[\sigma_{N,\mu}\cdot\nu_E]_E\|_{\mathcal{E}(T)}^2\right).$$

For Galerkin projection $u_N \in \prod_{\mu\in\Lambda} v_P(\mathcal{T})$

$$\|u_N - u_\Lambda\|_A \approx \|r_\Lambda(u_N)\|_{A^*} \lesssim \eta(u_N,\Lambda,\mathcal{T}) := \left(\sum_{T\in\mathcal{T}} \eta_T(u_N,\Lambda)^2\right)^{1/2}.$$

Combined upper error bound [EGSZ1]

$$\|u_N - u\|_A^2 \approx \|r(u_N)\|_{A^*}^2 \lesssim \eta(u_N, \Lambda, \mathcal{T})^2 + \zeta(u_N, \partial\Lambda)^2$$

adaptive algorithm

- evaluate Galerkin solution u_N
- evaluate error bounds $\eta(u_N, \Lambda, \mathcal{T})$ and $\zeta(u_N, \partial\Lambda)$
- spatial refinement if η dominates
- stochastic refinement if ζ dominates
- enlarge active set Λ

Let $V_\mu = V_p(\mathcal{T}_\mu) \subset H_0^1(D)$ for $\mu \in \Lambda$ and some simplicial mesh $\mathcal{T}_\mu$ of D.

Sparse approximation

$$u_N(y,x) = \sum_{\mu\in\mathcal{F}} u_{N,\mu}(x) P_\mu(y), \quad u_N = (u_{N,\mu})_{\mu\in\mathcal{F}} \in \prod_{\mu\in\mathcal{F}} V_\mu.$$

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Sparse approximation

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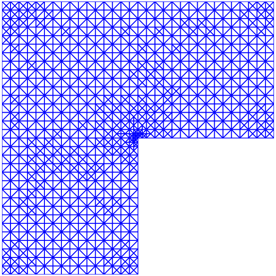
Coupling requires projections between compatible FE spaces

- for any $T \in \mathcal{T}_\mu$ and $T' \in \mathcal{T}_{\mu'}$, $T \cap T' \in \{\emptyset, T, T'\}$
- uniform local polynomial degree p
- localisation of projection errors for $T \in \mathcal{T}_\mu$, $\mu \in \Lambda$,

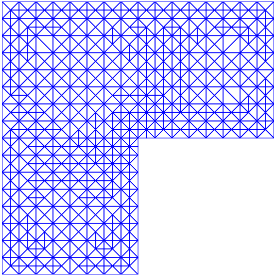
$$\zeta_{\mu,T}(u_N) := \sum_{m=1}^{\infty} \left\| \frac{a_m}{a_0} \right\|_{L^\infty(D)} \Big(\beta_{\mu_m+1} |u_{N,\mu+\epsilon_m} - \Pi_\mu u_{N,\mu+\epsilon_m}|_{H^1(T)} + \beta_{\mu_m} |u_{N,\mu-\epsilon_m} - \Pi_\mu u_{N,\mu-\epsilon_m}|_{H^1(T)} \Big).$$

L-shape meshes

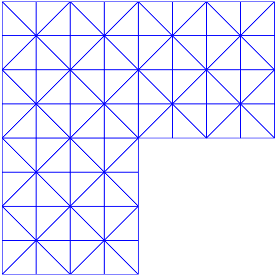
mesh [] (iteration 15)



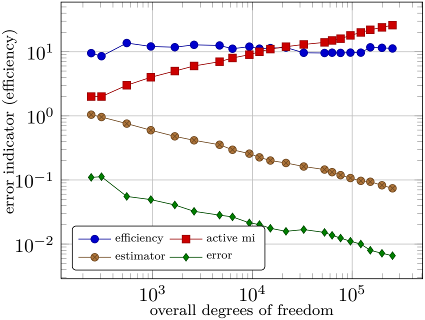
mesh [0001] (iteration 15)



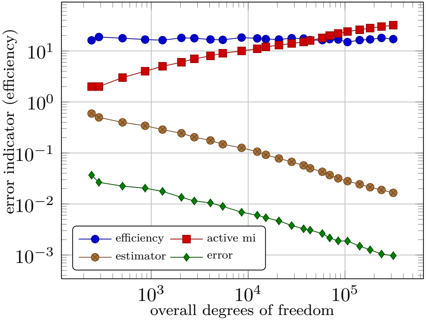
mesh [000001] (iteration 15)



L-shape and square efficiency



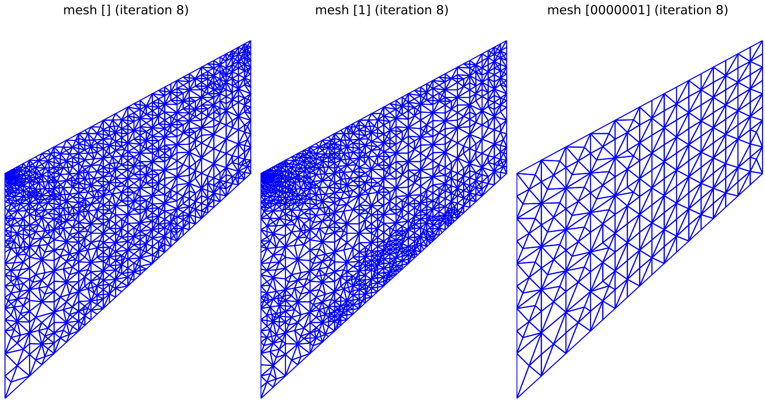
L-shape (slow decay)



square (slow decay)

- $a_m(x, y) := m^\alpha \cos(\pi\beta_1(m)x) \cos(\pi\beta_2(m)y))$
- scaled s.t. $\sum_{m=1}^{\infty} a_m = 9/10$
- $\alpha = -2$ (slow decay) and $\alpha = -4$ (fast decay)

Cook’s membrane meshes



Employ

- uniform local polynomial degree $p > 1$
- single mesh $\mathcal{T}$ for all active indices $\mu \in \Lambda$
- no projection errors among active coefficients

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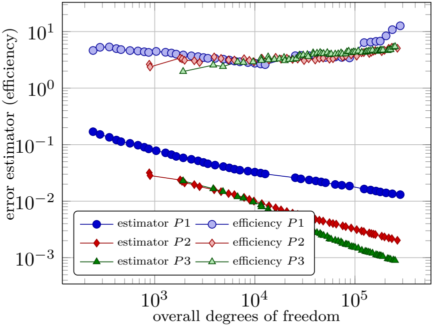
Adaptive algorithm is provably convergent [EGSZ2] since the quasi-error

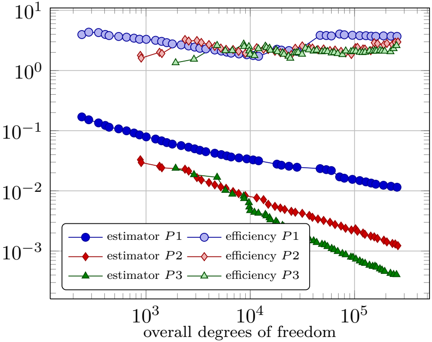
$$\|u_n - u\|_A^2 + \xi\eta(u_N, \Lambda, \mathcal{T})^2 + \omega\zeta(u_N, \partial\Lambda)$$

is a contraction w.r.t. combined stochastic/spatial refinement.

Numerical example (square, efficiency)

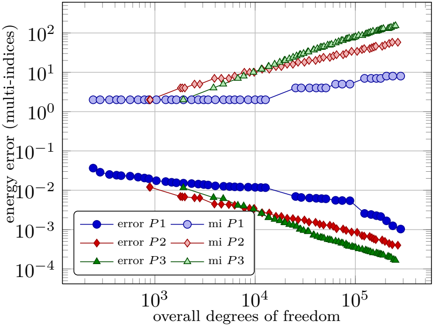


square (slow decay)

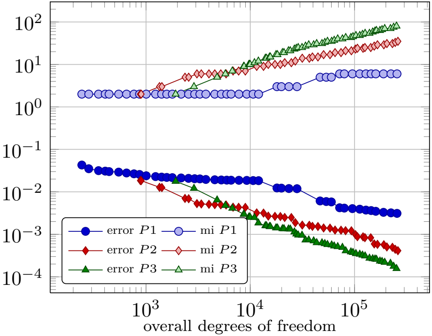


square (fast decay)

Numerical example (square, active mi)

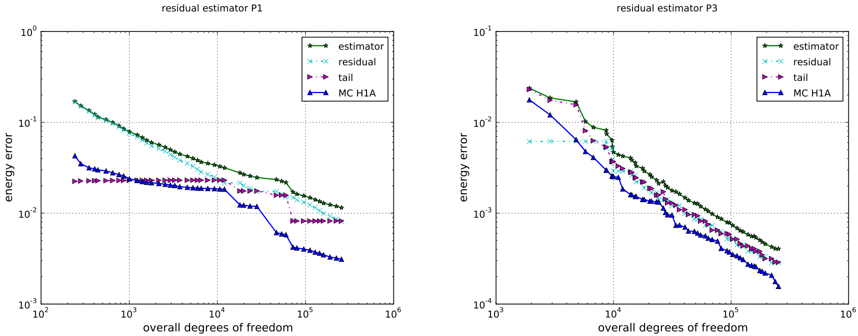


square (slow decay)



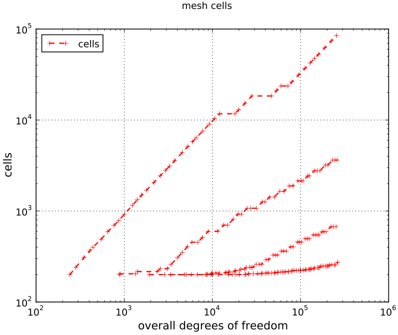
square (fast decay)

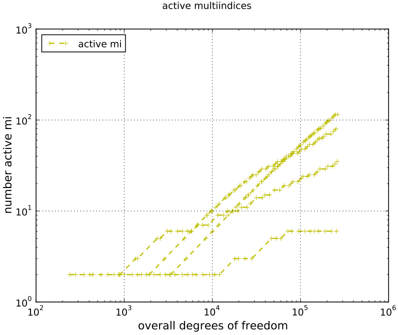
Numerical example (square, P1/P3 estimators)



error estimator and error: P1 (left) and P3 (right)

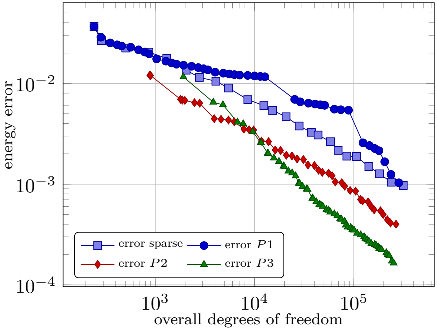
Numerical example (square, mesh/mi)

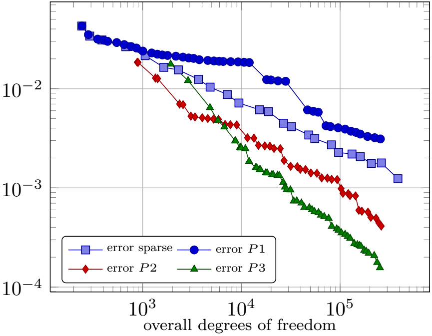




mesh sizes (left) and active multiindices (right)

Comparison of methods (square)





method I (sparse) vs. method II (higher order)

For approximation $u_N \in \mathcal{V}_N$ of u recall residual

$$\mathcal{R}\mathrm{es}(v) := \int_D f v - \int_D \sigma_N \cdot \nabla v \, dx$$

with discrete flux $\sigma_N := a \nabla u_N$. It holds

$$\|u - u_N\|_A \approx \|\mathcal{R}\mathrm{es}\|_{A^*}.$$

Any $q \in H(\mathrm{div}, D)$ yields

$$\|\mathcal{R}\mathrm{es}\|_{A^*} = \sup_{\substack{v \in \mathcal{V} \\ \|v\|_A = 1}} \int_D (f + \nabla \cdot q) v \, dx + \int_D (\sigma_N - q) \cdot \nabla v \, dx.$$

Different methods available to construct $q \in H(\mathbf{div}, D)$ s.t.

$$\int_T (f + \nabla \cdot q) v \, dx \leq \underbrace{C_{P,T}\, \mathbf{osc}_{T,q}}_{=:\widetilde{\mathbf{osc}}_{T,q}} \|\nabla v\|_{L^2(T)},$$

e.g. for $\int_T \nabla \cdot q + f_{\mathcal{T}} \, dx = 0$.

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e.g. for $\int_T \nabla \cdot q + f_{\mathcal{T}} \, dx = 0$. Then,

$$\|\mathcal{R}\mathrm{es}\|_{A^*} = \sup_{\substack{v \in \mathcal{V} \\ \|v\|_A = 1}} \sum_{T \in \mathcal{T}} \int_T (f + \nabla \cdot q) v \, dx + \int_T (q - \sigma_N) \cdot v \, dx$$

$$\le \left(\sum_{T \in \mathcal{T}} \Big(\widetilde{\mathbf{osc}}_{T,q} + \underbrace{\|a^{-1/2}(\sigma_N - q)\|_{L^2(T)}}_{=:\eta_T(q)} \Big)^2 \right)^{1/2}$$

In stochastic setting

- determine $q_\nu \in H(\mathbf{div}, D)$ with $\int_T \nabla \cdot q_\nu + f_{\mathcal{T}} \delta_{\nu 0} \, dx = 0$, $\nu \in \Lambda$
- it then holds

$$\|r_\nu(u_N)\|_{A^*}^2 \leq \sum_{T \in \mathcal{T}} \left(\|a_0^{-1/2}(q_\nu - \sigma_{N,\nu})\|_{L^2(T)} + \widetilde{\mathrm{osc}}_{\mathcal{T}, q_\nu} \right)^2$$

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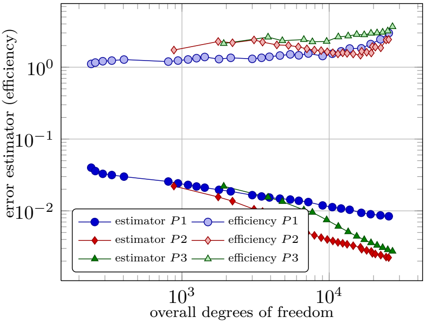
- it then holds

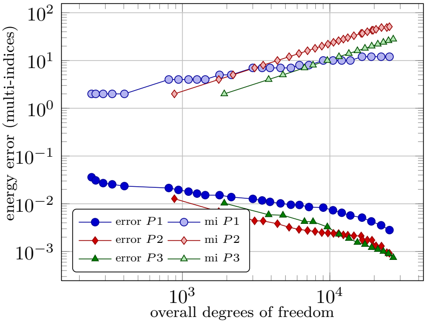
$$\|r_\nu(u_N)\|_{A^*}^2 \leq \sum_{T \in \mathcal{T}} \left(\|a_0^{-1/2}(q_\nu - \sigma_{N,\nu})\|_{L^2(T)} + \widetilde{\mathrm{osc}}_{\mathcal{T},q_\nu} \right)^2$$

- global minimisation for any discrete space $Q(\mathcal{T}) \subset H(\mathbf{div}, D)$

$$q_\nu = \mathbf{argmin}_{\tau \in Q(\mathcal{T})} \left\{ \|a_0^{-1/2}(\tau - \sigma_{N,\nu})\|_{L^2(D)} \right\}$$

e.g. $\mathrm{RT}_k(\mathcal{T})$ or $\mathrm{BDM}_k(\mathcal{T})$ with order k equal or greater than polynomial order of discrete flux $\sigma_{N,\nu} \in P_k(\mathcal{T}; \mathbb{R}^d)$.





square example, slow decay

Available results

- fully adaptive algorithms in spatial and stochastic variables
- construction of single-level or multilevel approximations
- based on techniques of adaptive FEM
- higher-order competitive with sparse approximations
- simulations carried out in open source framework `ALEA`

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Current development

- guaranteed a posteriori error est., local equilibration [EM]
- ASGFEM with low-rank tensor approximation [EZ]

References

Eigel, Gittelson, Schwab and Zander, *Adaptive stochastic Galerkin FEM*, SAM Report 2013–1, accepted.

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